

THERMAL STRESSES IN THIN SHALLOW SHELLS

BY

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DEPARTMENT OF CIVIL ENGINEERING

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THERMAL STRESSES IN THIN SHALLOW SHELLS

A THESIS SUBMITTED
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RAM SATTWAN

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CERTIFICATE

Certified that this work on "Thermal stresses in thin shallow shells" by Ram Sajwan, as a partial fulfilment towards the degree of Master of Technology in Civil Engineering, has been carried out under my supervision and that this has not been submitted elsewhere for a degree.



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SYNOPSIS

The present work involves the study of thermal stresses in thin shallow shells. Chapter I gives the introduction about the thermal stresses and amount and type of work done in thin shallow shells. Chapter II deals with the variation of total potential energy in deriving the basic equations for general thin shells and possible boundary conditions in thin shells. Chapter III deals with the making of nonhomogeneous boundary conditions to homogeneous conditions by Mindlin and Goodman technique. This chapter gives the solutions for two parallel edges simply supported and two other edges supported in any manner. Solutions have been given for (a) all four edges simply supported and (b) two parallel edges simply supported and remaining two edges clamped, where constants have been obtained with the help of Heung's Technique. An example illustrates the method.

A flow chart and a computer programme for (a) all edges simply supported and (b) two parallel edges simply supported and remaining two edges clamped, are given in Appendix (B) and (C) respectively. Tables and graphs are presented for deflections and moments in terms of dimensionless parameters for a/b ratio from 1.0 to 2.0 at interval of 0.2 and for a^2/hR_1 ratio from 5.0 to 20.0 at interval of 5.0 for elliptic paraboloids.

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NOTATIONS

A_1, A_2	Lame's constants
c_p	specific heat at constant deformation
d/dy (operator)	
E_1	$\frac{Eh^3}{12(1-\nu^2)}$ flexural rigidity of shell
E	Modulus of elasticity in tension or compression
ν	Poisson's ratio
λ	Thermal conductivity of the solid
M_1, M_2, M_{12}	Bending and twisting moments per unit distance in shell.
δ_T	$\text{at } z = h/2 \text{ T.z. } dz$ $-h/2$
δ_T	$\text{at } z = h/2 \text{ T. } dz$ $-h/2$
q_1, q_2, q_n	Components of intensity of surface loading
T	Temperature rise.
F_1, F_2, S	Normal and shearing forces per unit distance in middle surface of shell.
u, v, w	Components of displacements at distance z from middle plane.
α	Coefficient of linear expansion
β_1, β_2, α	Curvilinear Coordinates
$\sigma_1, \sigma_2, \sigma_3$	Normal components of stress parallel to β_1, β_2 and z -axis
$\gamma_{12}, \gamma_{13}, \gamma_{23}$	Shearing stress components in curvilinear coordinates.

$\epsilon_1, \epsilon_2, \epsilon_3$	Unit elongations in $\mathbf{e}_1, \mathbf{e}_2$, and \mathbf{e}_3 -directions.
$\epsilon_1^0, \epsilon_2^0$	Unit elongations in $\mathbf{e}_1, \mathbf{e}_2$ directions on the middle surface.
$\gamma_{12}, \gamma_{23}, \gamma_{13}$	Shearing strain components in curvilinear coordinates.
ψ	Stress function
ρ	Mass density
λ, μ	Lamé's constants
$\nabla^2(\psi)$	$\frac{1}{\lambda_1 \lambda_2} \left\{ \frac{1}{\epsilon_1} \left[\frac{\lambda_2^2}{\lambda_1} \frac{\psi_{,11}}{\psi_{,1}} \right] + \frac{1}{\epsilon_2} \left[\frac{\lambda_1^2}{\lambda_2} \frac{\psi_{,22}}{\psi_{,2}} \right] \right\}$
$I(\psi)$	$\frac{1}{\lambda_1 \lambda_2} \left\{ \frac{1}{\epsilon_1} \left[\frac{1}{\epsilon_2} \frac{\lambda_2^2}{\lambda_1} \frac{\psi_{,11}}{\psi_{,1}} \right] + \frac{1}{\epsilon_2} \left[\frac{1}{\epsilon_1} \frac{\lambda_1^2}{\lambda_2} \frac{\psi_{,22}}{\psi_{,2}} \right] \right\}$

CHAPTER I

INTRODUCTION

In modern design, there is a lot of attention towards the thermal stresses chiefly because of many engineering components which fail because of it. Thermal stresses play an important role in the design of gas or steam turbines, super sonic aircrafts, nuclear reactors, rocket engines, missiles, airframe structures, high-temperature oilrefining plants and structures operating at elevated temperatures.

Thermal stresses are arising from temperature effects. When temperature gradient is applied to material of the body, the various fibers tend to expand different amounts. To enable the body to remain continuous, a system of thermal strains and associated thermal stresses may be introduced which depend on the shape of the body and the temperature distribution inside the body. Thermal stresses are also introduced in cases where no temperature gradient exists if the part consists of several materials having different coefficient of expansions, or if free contraction or expansion is prevented by external constraints. As an example, a free plate having a temperature distribution varying with depth only will bend into a curved plate. Thermal stresses will be introduced in a free plate made of two materials, having different coefficient of expansions, with uniform temperature distribution. And thermal stresses are

also set in constrained edges plate having uniform temperature distribution.

The basic energy equation (1)^{**} derived from 'Law of conservation of energy' under the action of heat is given as

$$K T_{\text{eff}} = \rho C_E T + (3\lambda + 2\alpha) \Delta T_0 \quad \text{Eq. 1}$$

This is a coupled equation, which required simultaneous determination of temperature distribution and deformations.

This boundary value problem has considerable mathematical difficulty, since it combines the theories of heat conduction and elasticity under transient conditions. But fortunately in most of engineering problems, it is possible to introduce certain simplifying assumptions without significant error. The principal simplifying assumption is omission of mechanical coupling term in energy equation and giving rise some form of Fourier heat conduction equation, which is independent of stresses and deformations. If variations of strain in a body are produced by some external agency, these variations of strain are accompanied by variations in temperature and consequently flow of heat. This gives rise to an increase of entropy and corresponding increase of energy stored in a mechanically irrecoverable manner and this is thermoelastic dissipation, which requires for its study the use of coupled heat equation. The deformations due to external loads are accompanied by only small changes in temperature. It is then reasonable to calculate these deformations without taking account of

*Numbers refer to entries in the list of references.

thermal expansion. Similarly, if strains are produced in a body by non-uniform temperature distribution, it is clear that influence of these strains on the temperature itself is not too large. Thus omission of mechanical coupling term in energy equation would be meaningless.

Second simplifying assumption is omission of inertia term in an equation of motion. Duhamel had shown that the time rate of change of temperature is slow enough so that this term should not be significant. Danilevskaya (2) and Boley have also showed that omission of inertia term is meaningless. The two simplifying assumptions give rise to uncoupled quasi-static theory. This uncoupled quasi-static theory is divided into two distinct subjects, and known as theories of heat conduction and thermoelasticity. Solution of heat conduction equation gives the distribution of temperature inside the body.

In the present case, it is assumed that temperature distribution inside the thin shallow shell is known before hand. It is taken that temperature is anti-symmetric about the middle plane of the thin shallow shell but otherwise arbitrary.

Due to elastic constants such as Young's Modulus,

The thermal stresses are related in a complicated manner to the temperature distribution. Mechanical and Metallurgical properties may be changed with the temperature. It is assumed that elastic constants do not depend upon the temperature and material behaves elastically at all times.

Only Gradowczyk⁽³⁾ has tried to derive the governing equations for thin shallow shells. He has taken Navier type solution for deflection of thin shallow shell. Navier type solution is possible only for all edges simply supported. But Navier type solution is in terms of double trigonometric series which behaves poorly near the end point of interval. Gradowczyk has not solved completely for any type of boundary conditions.

Stress function and deflection of thin shallow shell subjected to temperature must satisfy the following two equations.

$$D_1 \nabla^2 \nabla^2 W - L(\theta) = q_0 - \nabla^2 \left(\frac{N_T}{1 - \nu^2} \right) \dots \dots (2)$$

and

$$\nabla^2 \nabla^2 \theta + E h L(W) = \nabla^2 N_T \dots \dots (3)$$

Stress and deflection functions are separated into different equations. Solutions of these equations

are obtained as Levy's type solutions which are in terms of hyperbolic and trigonometric functions. It gives better convergence near the end in comparison to Navier type solutions which is in terms of double trigonometric series.

The boundary conditions considered are:

- a. Two opposite edges of thin shallow shell simply supported and remaining two edges supported in any manner.
- b. All four edges of thin shallow shell simply supported.
- c. Two opposite edges of thin shallow shell simply supported and remaining two edges clamped.

CHAPTER II

FORMULATION OF BASIC EQUATIONS

Following assumptions have been made in the derivation of basic equations:

1. The material is homogeneous, isotropic and obeys Hooke's Law.
2. Planes perpendicular to middle surface before bending, remain plane and perpendicular to middle surface after bending and do not stretch in their length.

i.e. $\gamma_{13} = 0$, $\gamma_{23} = 0$, $\epsilon_3 = 0$

3. Normal stresses on planes parallel to middle surface are small in comparison to other stresses and these normal stresses are neglected.

i.e. $\sigma_3 = 0$

4. Deflections are small compared to thickness and the effect of loads in the plane of shell is neglected and coordinates of a particle before and after deformation are the same.

5. The elastic constants E , ν and the coefficient of thermal expansion are independent of temperature.

6. The thickness of the thin shell is very small in comparison with the least radius of curvature of the middle surface.

i.e. $1 \frac{1}{2} z/R_1 = 1$ and $1 \frac{1}{2} z/R_2 = 1$

7. Thickness of the shell in uniform.

Let μ_1 and μ_2 be the coordinate of point on the middle surface of thin shell and s be the distance measured along the axis and normal to the middle surface. The element of the thin shell acted by different forces is shown in Figure 1. Interpolation approximations for the displacements (ψ) as follows:

$$+\pi v_0(\beta_1, \beta_2) + \pi \dagger \quad \quad \quad \text{.....(4b)}$$

where u_0 and v_0 are the components of displacement at the middle surface; ϕ and ψ are the change of slopes of the normals to the middle surface.

Define following stress resultant values. Invent
of rest by reexamination.

$\frac{h}{2}$	$\frac{h}{2}$
$z_1 = \int_{-h/2}^{h/2} \sigma_1 z \, dz$	$u_1 = \int_{-h/2}^{h/2} \sigma_1 z \, dz$
$\frac{h}{2}$	$\frac{h}{2}$
$z_2 = \int_{-h/2}^{h/2} \sigma_2 z \, dz$	$u_2 = \int_{-h/2}^{h/2} \sigma_2 z \, dz$
$\frac{h}{2}$	$\frac{h}{2}$
$z = \int_{-h/2}^{h/2} \tau_{12} z \, dz$	$u_{12} = \int_{-h/2}^{h/2} \tau_{12} z \, dz$
$\frac{h}{2}$	$\frac{h}{2}$
$z_1 = \pm \int_{-h/2}^{h/2} 1 \, dz$	$u_1 = \pm \int_{-h/2}^{h/2} 1 \, dz$

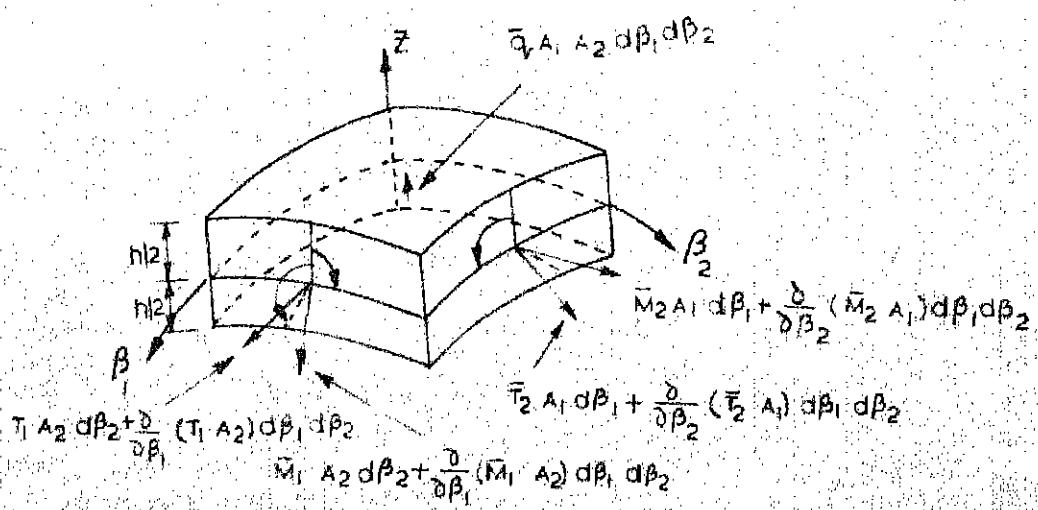


FIG. 1 ELEMENT OF SHELL AT EQUILIBRIUM

transient solutions are modified to take into account the changes due to temperature and other basic equations of transductivity are same as the ordinary thin shell theory. (2)

$$T_2 = \frac{4}{(T-u^2)} \quad (6_1 + u \cdot 6_2 = (1+u) \otimes 2) \quad \text{.....(60)}$$

$$T_{12} = \frac{Y}{g(1+y)} \quad \text{.....(62)}$$

strain components at any point at a distance a from the middle surface of shell in terms of middle plane strain components and curvatures are given by

$$g_{11} = g_{11}^0 + \pi E_1 \quad \quad \quad \gamma_{11} = 0$$

$$E_3 = E_2 + \epsilon E_2 \quad \gamma_{23} = 0 \quad \epsilon(g=0)$$

$$Y_{12} = Y_{12}^0 + \alpha K_{12} \quad \quad \quad \epsilon_3 = 0 \quad \quad \quad (3b)$$

$$R_1 = \frac{v_0}{A_1 A_2} - \frac{\partial \epsilon_1}{\partial \theta_2} + \frac{w}{A_1} + \frac{1}{A_1} - \frac{w_0}{\theta_1} \quad \quad \quad (3c)$$

$$\epsilon_2^0 = \frac{v_0}{A_1 A_2} \frac{\partial}{\partial \theta_1} A_2 + \frac{w}{A_2} + \frac{1}{A_2} - \frac{w}{\theta_2} \quad \quad \quad (3d)$$

$$Y_{12}^0 = \frac{1}{A_1} \frac{\partial}{\partial \theta_1} \left(- \frac{v_0}{A_2} + \frac{1}{A_2} \frac{\partial w}{\partial \theta_2} \right) + \frac{A_1}{A_2} \frac{\partial}{\partial \theta_2} \left(- \frac{w_0}{A_1} \right) \quad \quad \quad (3e)$$

$$k_1 = - \frac{1}{A_1} \frac{\partial}{\partial \theta_1} \left(- \frac{v_0}{A_2} + \frac{1}{A_2} \frac{\partial w}{\partial \theta_2} \right) - \frac{1}{A_1 A_2} \left(- \frac{v_0}{A_1} \right) \quad \quad \quad (3f)$$

$$k_2 = - \frac{1}{A_2} \frac{\partial}{\partial \theta_2} \left(- \frac{v_0}{A_1} + \frac{1}{A_1} \frac{\partial w}{\partial \theta_1} \right) - \frac{1}{A_1 A_2} \left(- \frac{w_0}{A_2} \right) \quad \quad \quad (3g)$$

$$k_{12} = \frac{A_1}{A_2} \frac{\partial}{\partial \theta_1} \left(- \frac{v_0}{A_2} + \frac{1}{A_2} \frac{\partial w}{\partial \theta_2} \right) - \frac{A_1}{A_2} \frac{\partial}{\partial \theta_2} \left(- \frac{w_0}{A_1} \right) \quad \quad \quad (3h)$$

Substituting Eq. (6) into Eq. (5), the stress and stress couples in terms of the middle and curvature are obtained as

$$r_1 = \frac{2h}{1-\nu} (\epsilon_1^0 + \epsilon_2^0) - \frac{2h}{1-\nu} \quad \quad \quad (3i)$$

$$X_2 = \frac{\alpha_1}{(1-\nu)} (0 + \frac{0}{\alpha_1}) = \frac{\alpha_1}{(1-\nu)} \quad \text{***** (3b)}$$

$$X_1 = \frac{\alpha_2}{(1-\nu)} \quad \text{***** (3c)}$$

$$X_1 = \alpha_1 (K_1 + \frac{X_2}{\alpha_2}) = \frac{\alpha_1}{(1-\nu)} \quad \text{***** (3d)}$$

$$X_2 = \alpha_2 (K_2 + \frac{X_1}{\alpha_1}) = \frac{\alpha_2}{(1-\nu)} \quad \text{***** (3e)}$$

$$X_{12} = + \frac{\alpha_1}{2} (1-\nu) X_{12} \quad \text{***** (3f)}$$

$$q = \frac{\alpha_2}{\alpha_1} = \frac{1}{\alpha_1} = \frac{\partial w}{\partial \alpha_1} \quad \text{***** (3g)}$$

$$q = \frac{\alpha_2}{\alpha_1} = \frac{1}{\alpha_1} = \frac{\partial w}{\partial \alpha_1} \quad \text{***** (3h)}$$

stress force in terms of stress coupling are given by

$$X_1 = \frac{1}{A_1 A_2} (2 \frac{\partial A_2 X_{12}}{\partial \alpha_2} + \frac{\partial A_2 X_1}{\partial \alpha_1} - \frac{\partial A_2}{\partial \alpha_1} X_2) = \frac{1}{A_2} \frac{\partial X_{12}}{\partial \alpha_2} \quad (3a)$$

$$X_2 = \frac{1}{A_1 A_2} (2 \frac{\partial A_2 X_{12}}{\partial \alpha_1} + \frac{\partial A_1 X_2}{\partial \alpha_2} - \frac{\partial A_1}{\partial \alpha_2} X_1) = \frac{1}{A_1} \frac{\partial X_{12}}{\partial \alpha_1} \quad (3b)$$

Point equilibrium equations and equilibrium conditions for the location of an electric thin shell have been derived by minimizing the total potential energy Ψ as follows:

$$\begin{aligned}
 \mathbf{r} = & \frac{1}{2} \int \int \left(\sigma_1 \theta_1^2 + \sigma_2 \theta_2^2 + \sigma_3 \theta_3^2 + Y_{12} T_{12} + Y_{23} T_{23} \right. \\
 & \left. + Y_{13} T_{13} \right) A_1 A_2 d\theta_1 d\theta_2 dz = \int \int \left(\sigma_1 u_0 + \sigma_2 v_0 \right. \\
 & \left. + \sigma_3 w \right) A_1 A_2 d\theta_1 d\theta_2
 \end{aligned} \quad (10)$$

By putting the values of σ_1 , σ_2 , σ_3 , T_{12} , γ_{23} and γ_{13} in Eq. (15)

By putting the values of u_1 , u_2 , and u_3 in Eq. (11)

$$r = \frac{1}{2(1-\nu)^2} \left\{ (e_1^0 + e_2^0)^2 - 2(1-\nu) (e_1^0 e_2^0 - \frac{v_0^2}{4}) \right. \\ \left. + v^2 \left\{ (k_1 + k_2)^2 - 2(1-\nu) (k_1 k_2 - \frac{k^2}{4}) \right\} \right\}$$

$$= 2(1+\nu) \times 2 \left\{ \epsilon_1^0 + \epsilon_2^0 + \nu(\kappa_1 + \kappa_2) \right\} \{ A_1 A_2 \Delta \theta_1 \Delta \theta_2 \} \\ - \iint (q_1 u_0 + q_2 v_0 + qn w) A_1 A_2 d\theta_1 d\theta_2 \quad \dots \dots \dots \quad (12)$$

Integrating with respect to s

$$\nu = \frac{1}{2(1+\nu)^2} \iint \left\{ \nu \left\{ (\epsilon_1^0 + \epsilon_2^0)^2 - 2(1+\nu)(\epsilon_1^0 \epsilon_2^0 - \gamma_{12}^{0^2}) \right\} \right. \\ \left. + \frac{\partial n^2}{\partial s} \left\{ (\kappa_1 + \kappa_2)^2 - 2(1+\nu)(\kappa_1 \kappa_2 - \frac{K_{12}^2}{4}) \right\} \right\} \\ = 2(1+\nu) \left\{ \kappa_2 (\epsilon_1^0 + \epsilon_2^0) + \frac{1}{2} (\kappa_1 + \kappa_2) \right\} \{ A_1 A_2 \Delta \theta_1 \Delta \theta_2 \} \\ = \iint (q_1 u_0 + q_2 v_0 + qn w) A_1 A_2 d\theta_1 d\theta_2 \quad \dots \dots \dots \quad (13)$$

Minimizing the total potential energy 'V' by variational principle, we get

$$\delta V = \iint \kappa_1 \delta \epsilon_1^0 + \kappa_2 \delta \epsilon_2^0 + \nu \delta \gamma_{12}^0 + \kappa_1 \delta \kappa_1 + \kappa_2 \delta \kappa_2 \\ + \delta \kappa_{12} \delta \kappa_{12} - (q_1 \delta u_0 + q_2 \delta v_0 + qn \delta w) \{ A_1 A_2 \Delta \theta_1 \Delta \theta_2 \} \\ = 0 \quad \dots \dots \dots \quad (14)$$

Changes in curvature and twist caused by displacement components tangential to the middle surface are very small and neglected in κ_1 , κ_2 and κ_{12} .

Carrying out the integration by parts, Eq.(14) gives the expression

$$\delta V = \iint \left[\left\{ \frac{\partial A_2}{\partial \theta_1} \kappa_2 - \frac{\partial}{\partial \theta_1} (A_2 \kappa_1) - \frac{\partial A_1}{\partial \theta_2} \nu + \frac{\partial}{\partial \theta_2} (A_1 \nu) - A_1 A_2 \epsilon_1 \right\} \delta u_0 \right.$$

$$\begin{aligned}
 & + \left\{ \frac{\partial A_1}{\partial \beta_2} T_1 - \frac{\partial}{\partial \beta_2} (A_1 T_2) - \frac{\partial A_2}{\partial \beta_1} S - \frac{\partial}{\partial \beta_1} (A_2 S) - A_1 A_2 q_2 \right\} \delta v_0 \\
 & + \left\{ D_1 \nabla^2 \nabla^2 w + \frac{T_1}{R_1} + \frac{T_2}{R_2} + \nabla^2 \frac{M_2}{(1-\nu)} - q_n \right\} \delta w \} \delta \beta_1 \delta \beta_2 \\
 & + \int \left[A_1 \left\{ T_2 \delta v_0 + S \delta u_0 - \frac{M_2}{A_2} \frac{\partial \delta w}{\partial \beta_2} + \left(M_2 + \frac{1}{A_1} \frac{\partial A_2}{\partial \beta_1} \right) \delta w \right\} \right] \delta \beta_1 \\
 & + \int \left[A_2 \left\{ T_1 \delta u_0 + S v_0 - \frac{M_1}{A_1} \frac{\partial \delta w}{\partial \beta_1} + \left(M_1 + \frac{1}{A_2} \frac{\partial A_1}{\partial \beta_2} \right) \delta w \right\} \right] \delta \beta_2 \\
 & - 2 \left[M_{12} \delta w \right]_0^{\beta_1 \beta_2} \\
 & = 0 \quad \dots \dots \dots (15)
 \end{aligned}$$

In the above expression, equating the coefficients of δu_0 , δv_0 and δw equal to zero, gives the equilibrium equations.

$$\delta u_0 : \frac{\partial}{\partial \beta_1} (A_2 T_1) - \frac{\partial A_2}{\partial \beta_1} T_2 + \frac{\partial A_1}{\partial \beta_2} S + \frac{\partial}{\partial \beta_2} (A_1 S) + A_1 A_2 q_1 = 0 \quad \dots \dots \dots (16)$$

$$\delta v_0 : \frac{\partial}{\partial \beta_2} (A_1 T_2) - \frac{\partial A_1}{\partial \beta_2} T_1 + \frac{\partial A_2}{\partial \beta_1} S + \frac{\partial}{\partial \beta_1} (A_2 S) + A_1 A_2 q_2 = 0 \quad \dots \dots \dots (17)$$

$$\delta w : D_1 \nabla^2 \nabla^2 w + \frac{T_1}{R_1} + \frac{T_2}{R_2} = q_n - \nabla^2 \frac{M_2}{(1-\nu)} \quad \dots \dots \dots (18)$$

where,

$$\nabla^2 (\dots) = \frac{1}{K_1 K_2} \left\{ \frac{\partial}{\partial \beta_1} \left[\frac{K_2}{K_1} \frac{\partial (\dots)}{\partial \beta_1} \right] + \frac{\partial}{\partial \beta_2} \left[\frac{K_1}{K_2} \frac{\partial (\dots)}{\partial \beta_2} \right] \right\}$$

Vanishing of surface integral terms in equation (18) give the appropriate boundary conditions considering the surface integrals, to get the conditions along $\beta_1 = 0$ and $\beta_1 = \beta_1$, the surface integral must vanish.

$$\int A_2 \left\{ T_1 \delta u_0 + S \delta v_0 - \frac{M}{K_1} \frac{\partial \delta w}{\partial \beta_1} + \left(U_1 + \frac{1}{K_2} \frac{\partial M}{\partial \beta_2} \right) \delta w \right\} d\beta_2 = 0 \quad (19-a)$$

2 This can be done by choosing

$$\begin{aligned} T_1 \delta u_0 &= 0 ; & S \delta v_0 &= 0 & (19-b) \\ M \frac{\partial \delta w}{\partial \beta_1} &= 0 ; & \left(U_1 + \frac{1}{K_2} \frac{\partial M}{\partial \beta_2} \right) \delta w &= 0 \end{aligned}$$

where $U_1 + \frac{1}{K_2} \frac{\partial M}{\partial \beta_2}$ is called Kirchhoff's shear.

This physically means that the virtual work done by forces and moments along the boundaries $\beta_1 = 0$ and $\beta_1 = \beta_1$ must be zero. This can be done in three ways, firstly by making all forces and moments to be zero along the boundary or secondly by choosing the displacements and slopes along the boundary to be zero, or thirdly by choosing partly forces and moments to be zero and partly displacements and slopes to be zero along the boundary.

Similarly vanishing of surface integral for $\beta_2 = 0$ and $\beta_2 = \beta_2$ is given

$$\int A_1 \{ T_2 \delta v_0 + S \delta u_0 - \frac{M_2}{A_2} \frac{\partial \delta w}{\partial \beta_2} + (X_2 + \frac{1}{A_1} \frac{\partial M_2}{\partial \beta_1}) \delta w \} d\beta_1 = 0 \quad (20-a)$$

Boundary conditions along $\beta_2 = 0$ and $\beta_2 = \beta_2$ can be obtained as

$$\begin{aligned} T_2 \delta v_0 &= 0 & S \delta u_0 &= 0 \\ M_2 \frac{\partial \delta w}{\partial \beta_2} &= 0 & (X_2 + \frac{1}{A_1} \frac{\partial M_2}{\partial \beta_1}) \delta w &= 0 \end{aligned} \quad (20-b)$$

At the four corners,

$$2M_{12} \delta w = 0 \quad (20-c)$$

Consider a stress function ' θ ' which is related to the inplane forces as follows:

$$T_1 = -\frac{1}{A_2} \frac{\partial}{\partial \beta_2} \left(\frac{1}{A_2} \frac{\partial \theta}{\partial \beta_2} \right) = -\frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \beta_1} \frac{\partial \theta}{\partial \beta_1} \quad (21-a)$$

$$T_2 = -\frac{1}{A_1} \frac{\partial}{\partial \beta_1} \left(\frac{1}{A_1} \frac{\partial \theta}{\partial \beta_1} \right) = -\frac{1}{A_2 A_1} \frac{\partial A_1}{\partial \beta_2} \frac{\partial \theta}{\partial \beta_2} \quad (21-b)$$

$$S = \frac{1}{A_1 A_2} \left(\frac{\partial^2 \theta}{\partial \beta_1 \partial \beta_2} - \frac{1}{A_1} \frac{\partial A_1}{\partial \beta_2} \frac{\partial \theta}{\partial \beta_1} - \frac{1}{A_2} \frac{\partial A_2}{\partial \beta_1} \frac{\partial \theta}{\partial \beta_2} \right) \quad (21-c)$$

Putting the values of T_1 , T_2 and S into eqns. (16), (17) and (18) and simplifying by using the Gauss-Codazzi relations⁽⁵⁾.

$$\frac{1}{A_1 A_2} \frac{1}{A_1} \frac{\partial \theta}{\partial \beta_1} = q_1 \quad (22)$$

$$\frac{1}{A_1 A_2} \frac{1}{A_2} \frac{\partial \theta}{\partial \beta_2} = q_2 \quad (23)$$

$$B_1 \nabla^2 \nabla^2 w - \Sigma(\sigma) = q_n - \nabla^2 \frac{\frac{E_1}{1-\nu}}{(1-\nu)} \quad (24)$$

$$\Sigma(\dots) = \frac{1}{E_1 A_2} \left\{ \frac{\partial}{\partial \rho_1} \left[\frac{1}{R_2} \frac{A_2}{A_1} \frac{\partial (e_{12})}{\partial \rho_1} \right] + \right. \\ \left. + \frac{\partial}{\partial \rho_2} \left[\frac{1}{R_1} \frac{A_1}{A_2} \frac{\partial (e_{12})}{\partial \rho_2} \right] \right\}$$

In absence of forces in ρ_1 and ρ_2 directions, Eqns. (22) and (23) will be automatically satisfied for shell of zero Gaussian curvature, for shell of non zero Gaussian curvature, the above equations are approximately satisfied for shallow shells. In order to obtain, the unknown stress function ' σ ', now we have to consider the compatibility conditions.

Compatibility Equation is given by

$$\frac{k_1}{R_2} + \frac{k_2}{R_1} + \frac{1}{A_1 A_2} \left[\frac{\partial}{\partial \rho_1} \frac{1}{A_1} \left\{ A_2 \frac{\partial e_1^0}{\partial \rho_1} + \frac{\partial A_2}{\partial \rho_1} (e_2^0 - e_1^0) - \right. \right. \\ \left. \left. \frac{1}{2} A_1 \frac{\partial e_{12}^0}{\partial \rho_2} - \frac{\partial A_1}{\partial \rho_2} e_{12}^0 \right\} + \frac{\partial}{\partial \rho_2} \frac{1}{A_2} \left\{ A_1 \frac{\partial e_2^0}{\partial \rho_2} + \right. \right. \\ \left. \left. \frac{\partial A_1}{\partial \rho_2} (e_1^0 - e_2^0) - \frac{1}{2} A_2 \frac{\partial e_{12}^0}{\partial \rho_1} - \frac{\partial A_2}{\partial \rho_1} e_{12}^0 \right\} \right] = 0 \quad (25)$$

By substituting the values of k_1 , k_2 , e_1^0 , e_2^0 , e_{12}^0 in Eqn. (25).

$$EhL(w) + \nabla^2 \nabla^2 \sigma + \frac{(1+\nu)}{E_1 A_2} \left[\frac{\partial}{\partial \rho_1} \left(\frac{1}{R_1 A_1} \frac{A_2}{R_2 \partial \rho_1} \right) + \right. \\ \left. \frac{\partial}{\partial \rho_2} \left(\frac{1}{R_1 A_2} \frac{A_1}{R_2 \partial \rho_2} \right) \right] = \nabla^2 u_T \quad (26)$$

for zero Gaussian curvature ($1/R_1 R_2 = 0$)

$$EhL(w) + \nabla^2 \nabla^2 \sigma = \nabla^2 u_T \quad (27)$$

These equations hold good for

1. shells of zero Gaussian curvature such as cylindrical shells and cones.
2. good for shallow shells
3. approximate for non-zero Gaussian curvature.

These equations are coupled and requires the simultaneous solution of deflection 'w' and stress function 'θ'. If there is no external loading, for constant thickness equation (24) reduces to the form,

$$D_1 \nabla^2 \nabla^2 w - L(\theta) = - \nabla^2 \frac{E_1}{(1-\nu)} \quad (28)$$

Superposition of deflections due to transverse load alone and deflection due to temperature distribution alone is possible. Solution of deflection due to temperature distribution alone has been utilised in the present case.

CHAPTER III

METHOD OF SOLUTIONS

In this chapter, solutions to obtain thermal stresses in thin shallow shells with two edges simply and other two edges supported in any manner are developed. The equations governing, the transverse deflection of elastic thin shells subjected to a temperature distribution which is anti-symmetric about the middle plane and otherwise arbitrary are:

$$D_1 \nabla^2 \nabla^2 w - L(\theta) + \nabla^2 \frac{M_T}{(1-\nu)} = 0 \quad (29)$$

$$EhL(w) + \nabla^2 \nabla^2 \theta - \nabla^2 M_T = 0 \quad (30)$$

Deflection function 'w' and stress function 'θ' may be separated out as

$$D_1 \nabla^6 \theta + EhL^2(\theta) = D_1 \nabla^6 M_T + Eh \nabla^2 L \left(\frac{M_T}{(1-\nu)} \right) \quad (31)$$

$$D_1 \nabla^6 w + EhL^2(w) = -\nabla^6 \frac{M_T}{(1-\nu)} + \nabla^2 L (M_T) \quad (32)$$

Both functions w and θ , have the same homogeneous solutions but differing in constants of integrations.

Doubly curved thin shells may be reduced to thin shallow shells, when it is satisfying the following conditions

$$z = z(x, y) \quad , \quad H < \frac{1}{3} (a, b)$$

$$\left(\frac{\partial z}{\partial x} \right)^2 \ll 1 \quad , \quad \left(\frac{\partial z}{\partial y} \right)^2 \ll 1, \quad \left(\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} \right) \ll 1$$

Thin shallow shell has been shown in fig. (2) with coordinate system. The value of A_1 and t_2 for shallow shell may be approximated as

$$A_1 = A_2 = 1, t_1 \text{ and } R_2 \text{ are constant.}$$

Stress couples and stress resultants for thin shallow shell in terms of deflection 'w' and stress function θ may be expressed as

$$N_x = -D_1 \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) = \frac{N_T}{(1-\nu)} \quad (33)$$

$$N_y = +D_1 \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) = \frac{N_T}{(1-\nu)} \quad (34)$$

$$N_{xy} = +D_1(1-\nu) \frac{\partial^2 w}{\partial x \partial y} \quad (35)$$

$$T_1 = -\frac{\partial^2 \theta}{\partial y^2}, \quad T_2 = -\frac{\partial^2 \theta}{\partial x^2}, \quad S = \frac{\partial^2 \theta}{\partial x \partial y} \quad (36-a-b)$$

The middle plane strains in terms of stress resultants may be expressed as,

$$\epsilon_1^0 = \frac{1}{Eh} (T_1 - \nu T_2 + N_T), \quad \epsilon_2^0 = \frac{1}{Eh} (T_2 - \nu T_1 + N_T) \quad (37-a-b)$$

Doubly curved thin shallow shell is simply supported along $x = 0$ and $x = a$ and making conditions such that

$$T_1(0) = 0 \text{ and } \epsilon_2^0(0) = 0 \text{ (using tie rod)}$$

$$T_1(a+y) = -\frac{\partial^2 \theta(a,y)}{\partial y^2} = 0$$

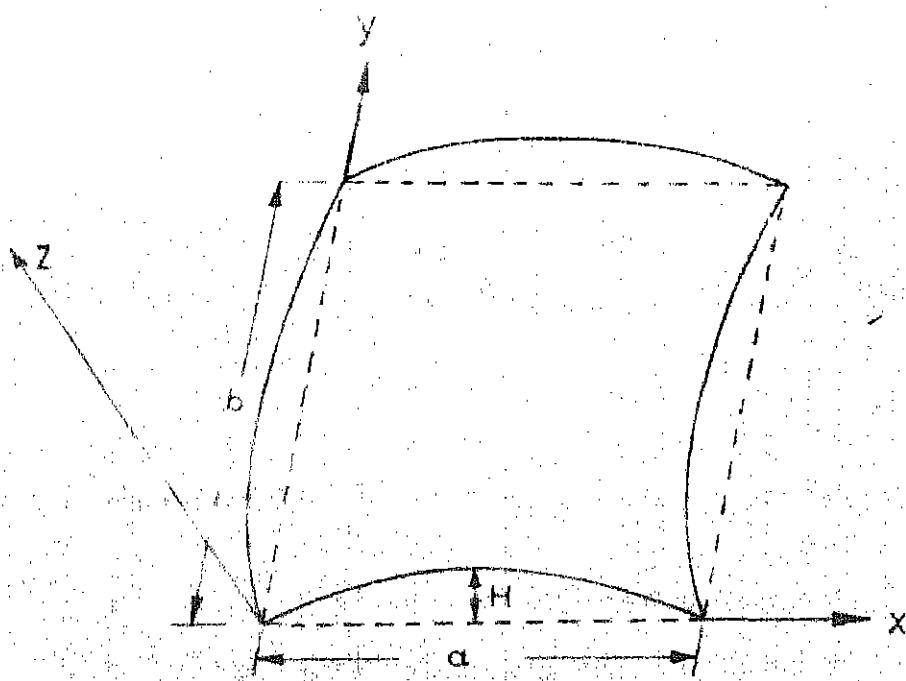


FIG.2 SHALLOW SHELL WITH COORDINATES

or

$$\theta(0, y) = 0 \quad (38)$$

Distribution of temperature is antisymmetric about the middle plane so that $\theta(0, y) = 0$

$$\frac{\partial^2 \theta}{\partial x^2} \Big|_{x=0, a} = 0$$

or

$$\frac{\partial^2 \theta}{\partial x^2} (0, y) = 0 \quad (39)$$

$$w = 0; \quad H_x = 0 \quad \text{when } x = 0, a \quad (40)$$

Boundary conditions (40) imply that

$$w = 0; \quad \frac{\partial^2 w}{\partial x^2} = -H_T(0, y)/D_T(1-\nu) \quad \text{along } x = 0$$

$$w = 0; \quad \frac{\partial^2 w}{\partial x^2} = -H_T(a, y)/D_T(1-\nu) \quad \text{along } x = a$$

The boundary conditions are non-homogeneous. The non-homogeneity is removed by taking the function 'w' into two forms^(6,7,8) as follows

$$w(x, y) = U(x, y) + \left\{ \frac{H_T(0, y)}{1-\nu} H_0(x) + \frac{H_T(a, y)}{1-\nu} H_a(x) \right\} \quad (41)$$

where $H_0(x)$ and $H_a(x)$ are the unknown functions of x only.

Applying the boundary conditions into the equation (41)

$$U(0, y) = - \left[\frac{H_T(0, y)}{1-\nu} H_0(0) + \frac{H_T(a, y)}{1-\nu} H_a(0) \right] \quad (42-a)$$

$$\frac{\partial^2}{\partial x^2} U(0, y) = - \frac{H_T(0, y)}{1-\nu} \left[\frac{1}{D_T} + \frac{d^2}{dx^2} \frac{H_0(0)}{1-\nu} \right] - \frac{H_T(a, y)}{1-\nu} \frac{d^2}{dx^2} \frac{H_a(0)}{1-\nu} \quad (42-b)$$

$$U(0, y) = - \left[\frac{H_0(0, y)}{C(1-\nu)} H_0(a) + \frac{H_a(0, y)}{C(1-\nu)} H_a(a) \right] \quad (42-c)$$

$$\frac{d^2}{dx^2} U(a, y) = - \frac{H_0(0, y)}{C(1-\nu)} \frac{d^2}{dx^2} H_0(a) - \frac{H_a(0, y)}{C(1-\nu)} \left[\frac{1}{D_1} + \frac{d^2}{dx^2} H_a(a) \right] \quad \dots (42-d)$$

Functions $H_0(x)$ and $H_a(x)$ are chosen in the following manner in order to make the equation (42) homogeneous.

$$H_0(0) = 0 \quad H_a(0) = 0$$

$$\frac{d^2}{dx^2} H_0(0) = - 1/D_1 \quad \frac{d^2}{dx^2} H_a(0) = 0 \quad (43-a) \quad (43-b)$$

$$H_0(a) = 0 \quad H_a(a) = 0$$

$$\frac{d^2}{dx^2} H_0(a) = 0 \quad \frac{d^2}{dx^2} H_a(a) = - 1/D_1$$

The functions $H_0(x)$ and $H_a(x)$ are taken as polynomials with four constants which are determined with above conditions on $H_0(x)$ and $H_a(x)$. They are of the form

$$H_0(x) = \frac{a^2}{D_1} \left[\frac{1}{3} \frac{x}{a} - \frac{1}{2} \frac{x^2}{a^2} + \frac{1}{6} \frac{x^3}{a^3} \right] \quad (44-a)$$

$$H_a(x) = \frac{a^2}{6D_1} \left[\frac{x}{a} - \frac{x^3}{a^3} \right] \quad (44-b)$$

The boundary conditions on $U(x, y)$ become,

$$U = 0, \quad \frac{d^2 U}{dx^2} = 0 \quad \text{along } x = 0 \text{ and } x = a.$$

Now substituting the eqn. (41) in eqn. (32)

$$D_1 \nabla^8 U + \pi h L^2 (U) = F(x, y) \quad (45-a)$$

where $F(x, y) = -\nabla^6 \frac{H_1}{(1-\nu)} + \nabla^2 L(H_1) = \pi \nabla^8 \left[\frac{H_1(0, y) H_0(x)}{(1-\nu)} \right. \\ \left. + \frac{H_2(x, y) H_0(x)}{(1-\nu)} \right] - \pi h L^2 \left[\frac{H_2(0, y) H_0(x)}{(1-\nu)} \right. \\ \left. + \frac{H_3(x, y) H_0(x)}{(1-\nu)} \right] \quad (45-b)$

Solution of eqn. (45-a) may be taken as

$$U(x, y) = \sum_{m=1}^{\infty} Y_m(y) \sin \frac{m\pi x}{a} \quad (46)$$

This form of solution will satisfy the boundary conditions along $x = 0$ and $x = a$.

Substituting the value of $U(x, y)$ in the equation 45(a) and simplifying

$$\left(-\frac{d^2}{dy^2} Y_m(y) - \frac{m^2 \pi^2}{a^2} Y_m(y) \right)^4 \\ + \frac{Eh}{D_1} \left(\frac{1}{R_1} \frac{d^2}{dy^2} Y_m(y) - \frac{1}{R_2} \frac{m^2 \pi^2}{a^2} Y_m(y) \right)^2 = K_m(y) \quad (47-a)$$

where $K_m(y) = \frac{1}{D_1} \frac{2}{a} \int_0^a f(x, y) \sin mx/a dx$

or $A_{1m} \frac{d^8}{dy^8} Y_m(y) + A_{2m} \frac{d^6}{dy^6} Y_m(y) + A_{3m} \frac{d^4}{dy^4} Y_m(y) \\ A_{4m} \frac{d^2}{dy^2} Y_m(y) + A_{5m} Y_m(y) = K_m(y) \quad (47-b)$

Solution of homogeneous equation:

$$A_{1m} \frac{d^8}{dy^8} Y_m(y) + A_{2m} \frac{d^6}{dy^6} Y_m(y) + A_{3m} \frac{d^4}{dy^4} Y_m(y) \\ A_{4m} \frac{d^2}{dy^2} Y_m(y) + A_{5m} Y_m(y) = 0 \quad (47-c)$$

Assume $Y_m(y)$ in the form:

$$Y_m(y) = C_m e^{\delta_m y/a}$$

Substituting value of $Y_m(y)$ into the equation (47-a), gives,

$$B_{1m} \frac{8}{m^2} + B_{2m} \frac{6}{m} + B_{3m} \frac{4}{m} + B_{4m} \frac{2}{m} + B_{5m} = 0 \quad (47-a)$$

where $B_{1m} = 1$ $A_m = B_{1m}$
 $B_{2m} = -4 m^2 \pi^2$ $A_{2m} = B_{2m}/a^2$
 $B_{3m} = (6 m^4 \pi^4 + T)$ $A_{3m} = B_{3m}/a^4$
 $B_{4m} = -2m^2 \pi^2 (2m^4 \pi^4 + TR)$ $A_{4m} = B_{4m}/a^6$ $(49-b)$
 $B_{5m} = m^4 \pi^4 (m^4 \pi^4 + R^2 T)$ $A_{5m} = B_{5m}/a^8$

$$T = 12(1-y)^2 \frac{a^4}{h^2 R_1^2}, \quad R = R_1/R_2$$

Roots of the equation (49) may be obtained by changing this equation into fourth order equation and solving by the method given in Ref. (9).

Depending upon values of A_{1m} to A_{5m} , eqn. (47-a) has different set of roots and different type of function $Y_m(y)$ as given in the Appendix (A).

The homogeneous solution⁽¹⁰⁾ of eqn. (47) may be expressed as

$$Y_m(y) = C_{1m} Y_{1m} + C_{2m} Y_{2m} + \dots + C_{5m} Y_{5m} \quad (50)$$

where Y_{1m} through Y_{5m} are the basis of solutions (Appendix A) and C_{1m} through C_{5m} are arbitrary constants to be evaluated from the boundary conditions of the edges of thin shallow shell

along $y = 0$ and $y = b$.

Writing the equation (50) in the matrix form as

$$Y_m(y) = [c]^T [Y_R(y)] \quad (51-1)$$

The superscript 'T' denotes the transpose of the matrix and

$$[c] = \begin{bmatrix} c_{1m} \\ c_{2m} \\ c_{3m} \\ c_{4m} \\ c_{5m} \\ c_{6m} \\ c_{7m} \\ c_{8m} \end{bmatrix} \quad [Y_R(y)] = \begin{bmatrix} y_{1m}(y) \\ y_{2m}(y) \\ y_{3m}(y) \\ y_{4m}(y) \\ y_{5m}(y) \\ y_{6m}(y) \\ y_{7m}(y) \\ y_{8m}(y) \end{bmatrix} \quad (51-b-c)$$

The differential equation of $[Y_R(y)]$ can be expressed as

$$\frac{d^n}{dy^n} [Y_R(y)] = [H_R]^n [Y_R(y)] \quad (51)$$

$$[H_R]^n = [H_R] [H_R] \dots \dots \text{ n Times} \quad (52)$$

where $[H_R]$ is a matrix depending on the characteristic roots of the differential equation (47) and is given in Appendix A.

$$\frac{d^n}{dy^n} Y_m(y) = [c]^T [H_R]^n [Y_R(y)] \quad (53)$$

From equation (51) it is clear that function θ has the identical homogeneous solution to the deflection function 'w' except integration constants.

For homogeneous boundary conditions at $x = 0$, and $x = a$

$$\theta(x, y) = \sum_{m=1,3}^{\infty} F_m(y) \sin mx/a \quad (54)$$

$$F_m(y) = [D]^T [Y_m(y)] \quad (55)$$

$$[D] = \begin{bmatrix} D_{1m} \\ D_{2m} \\ D_{3m} \\ D_{4m} \\ D_{5m} \\ D_{6m} \\ D_{7m} \\ D_{8m} \end{bmatrix}$$

There are sixteen unknowns, eight in deflection 'w' and eight in stress function 'θ'. But possible number of boundary conditions are eight along the edges $y = 0$ and $y = b$. Eight unknowns can be expressed in terms of other eight unknowns as given in reference (11).

Substitute the value of $\theta(x, y)$ and homogeneous boundary condition part of $w(x, y)$ in the equation (30)

$$\begin{aligned} \text{Eh} \left\{ -\frac{1}{E_2} \frac{\frac{w''''}{a^2} + \frac{1}{E_1} [u_R]^2}{a^2} \right\} [C]^T [Y_R(y)] \\ + \left\{ \frac{\frac{4}{E_2} \frac{4}{a^4} + [u_R]^4 - \frac{\frac{2}{E_1} \frac{2}{a^2} [u_R]^2}{a^2}}{a^2} \right\} [D]^T [Y_R(y)] = 0 \end{aligned} \quad \dots (56)$$

To satisfy this equation, equal coefficient of y_{1m} through y_{8m} equal to zero. This gives eight equations relating the constants $C_{1m}, C_{2m}, C_{3m}, C_{4m}, C_{5m}, C_{6m}, C_{7m}, C_{8m}$ to $D_{1m}, D_{2m}, D_{3m}, D_{4m}, D_{5m}, D_{6m}, D_{7m}, D_{8m}$. Expressing the D_{im} through D_{8m} constants into C_{1m} through C_{8m} . Thus C_{1m} through C_{8m} are known in terms of C_{1m} through C_{8m} . And there are eight unknowns C_{1m} through C_{8m} .

Solution of non-homogeneous equation by operators:

Equation (47-b) can be written as

$$(D-a_1)(D-a_2)\dots(D-a_8) = K(y) \quad (57)$$

where a_1, a_2 are the roots of the auxiliary equation

$$\text{Put } u_1 = (D-a_2)(D-a_3)\dots(D-a_8) Y_1(y)$$

so that equation (57) becomes

$$(D-a_1) u_1 = K(y)$$

This is a first-order linear equation in the unknown u_1 , for which a particular integral is readily found. Next put

$$u_2 = (D-a_3)(D-a_4)\dots(D-a_8) Y_2(y)$$

and write

$$(D-a_2) u_2 = u_1$$

where u_1 is known function of y , and solve this first order equation for u_2 continuing in this way, a sequence of linear first order equations are obtained.

$$(D-a_1) u_i = u_{i-1} \quad (i = 1, 2, \dots, 8) \quad (58)$$

where

$$u_0 = K(y), \quad u_1 = Y_m(y), \quad u_i = (D-a_{i-1})(D-a_{i+1}) \\ \dots (D-a_8) Y_m(y) \quad (i = 1, 2, \dots, 7)$$

and where, for each value of i , u_{i-1} is a known function of y .

After seven such steps, obtain the first first order differential equation of the sequence (58), namely

$$(D-a_8) Y_m(y) = u_7$$

a solution of which is a particular integral of (57). The sequential integration of the equation (58) is obviously a routine process. Each such equation has the integrating factor $e^{\int a_1 y} dy$ and the solution

$$u_i = e^{\int a_1 y} \int e^{-\int a_1 y} u_{i-1}(y) dy \quad (i = 1, \dots, 8)$$

The unknown constants C_{1m} through C_{8m} may be obtained by applying the suitable eight initial parameters, displacements, slopes, forces, and moments in terms of $Y_m(y)$, $Y_m'(y)$, $Y_m''(y)$, $r_m(y)$, $r_m'(y)$ and $r_m''(y)$ along the edges $y = 0$ and $y = 6$, depending upon the boundary conditions at $y = 0$ and $y = 6$.

This may be expressed in matrix form as

$$[I_{pm}] = [A] [C] \quad (59)$$

$$[C] = [A]^{-1} [I_{pm}] \quad (60)$$

The value of I_{pm} matrix depends upon the boundary conditions. Values of I_{pm} matrix have been given for simply supported along $y = 0$ and $y = b$ and clamped along $y = 0$ and $y = b$.

Matrix [A] is given as

$$[A] = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} & A_{18} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} & A_{27} & A_{28} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} & A_{37} & A_{38} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} & A_{47} & A_{48} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & A_{56} & A_{57} & A_{58} \\ A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} & A_{67} & A_{68} \\ A_{71} & A_{72} & A_{73} & A_{74} & A_{75} & A_{76} & A_{77} & A_{78} \\ A_{81} & A_{82} & A_{83} & A_{84} & A_{85} & A_{86} & A_{87} & A_{88} \end{bmatrix}$$

Value of Matrix [A] depends upon $[Y_p(y)]$ and type of boundary conditions along $y = 0$ and $y = b$.

Values of elements of Matrix [A] have been given in example for simply supported along $y = 0$ and $y = b$.

For the sake of simplicity, take temperature distribution independent of x and y Eqn. (47-b) reduces to homogeneous form. So deflection function may be given by

$$w(x, y) = \sum_{m=1}^{\infty} \left[Y_m(y) + 4 \frac{\frac{E}{1-\nu} \frac{a^2}{m^2 \pi^2}}{D_p(1-\nu) y^2 m^2} \right] \sin mx/\alpha$$

(a) Simply supported along the edges $x = 0$ and $y = b$:

The boundary conditions are,

$$w(x, \frac{b}{2}) = 0. \text{ This gives } Y_m(0) = Y_m(b) = -\frac{4E_p \frac{a^2}{m^2 \pi^2}}{D_p(1-\nu) b^2 m^2}$$

for m odd only

$$v_y(x, \frac{0}{b}) = 0$$

$$\text{This gives } \frac{d^2}{dy^2} Y_m(0) = \frac{a^2}{a^2} Y_m(0) = -\frac{4M_p}{(1-\nu) D_1 m}$$

for m odd only

$$v_x(x, \frac{0}{b}) = 0 \text{ this gives } Y_m(\frac{0}{b}) = 0$$

$$u(x, \frac{0}{b}) = 0 \text{ This gives } \frac{d^2}{dy^2} Y_m(\frac{0}{b}) = 0.$$

$$\begin{bmatrix} -\frac{4M_p a^2}{D_1(1-\nu) m^2 \pi^2} \\ -\frac{4M_p a^2}{D_1(1-\nu) m^2 \pi^2} \\ -4M_p / D_1(1-\nu) m \\ -4M_p / D_1(1-\nu) m \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) Fixed along the edges $y = 0$ and $y = b$

Boundary conditions are

$$v(x, \frac{0}{b}) = 0$$

$$\text{This gives } Y_m(0) = Y_m(b) = -\frac{4M_p a^2}{(1-\nu) D_1 m^2 \pi^2} \text{ for } m \text{ odd only}$$

$$\frac{\partial}{\partial y} W(x, \frac{y}{b}) = 0$$

$$\text{This gives } \frac{d}{dy} Y_R(0) = \frac{d}{dy} Y_R(b) = 0$$

$$x_2(x, \frac{y}{b}) = 0, \text{ this gives } F_R(\frac{y}{b}) = 0$$

$$U(x, \frac{y}{b}) = 0, \text{ this gives } \frac{d^2}{dy^2} x_R(\frac{y}{b}) = 0.$$

$$\begin{bmatrix} \frac{-4M_1 a^2}{D_1(1-\nu)m^3 \pi^3} \\ -4M_1 a^2/D_1(1-\nu)m^3 \pi^3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[X_{pm}] =$$

$$Y_R(y) = [C]^T [Y_R(y)]$$

$$W(x, y) = \sum_{m=1,3}^{\infty} \left\{ Y_R(y) + \frac{4M_1 a^2}{D_1(1-\nu)m^3 \pi^3} \right\} \sin mx/a \quad (61)$$

$$u_x = \sum_{m=1,3}^{\infty} \left\{ D_1 \frac{m^2 a^2}{a^2} [C]^T + [C]^T [Y_R]^2 \right\} [Y_R(y)] \sin mx/a \quad (62)$$

$$u_y = \sum_{m=1}^{\infty} \left\{ D_1 \left\{ -[C]^T [Y_R]^2 + \frac{m^2 a^2}{a^2} [C]^T \right\} [Y_R(y)] \right. \\ \left. - \frac{4M_1}{(1-\nu)m \pi} \right\} \sin mx/a \quad (63)$$

$$X_{xy} = + D_1 (1-\nu) \times \frac{\pi^2}{a} [C]^T [Y_R(y)] \cos \pi x/a \quad (64)$$

(c) Example:

In order to illustrate the method, an example with $\pi^2/4R_1=5$, $\nu = .3$, and $R_1/R_2 = 0.5$ has been worked out below.

Roots of the equation (49-a) have the form $\pm n_1, \pm n_2$
 $\pm (n_3 \pm in_4)$

$$Y_R(y)^T = \{ \cosh n_1 y/a, \sin n_1 y/a, \cosh n_2 y/a, \sinh n_2 y/a, \\ \cosh n_3 y/a \cos n_4 y/a, \sinh n_3 y/a \cos n_4 y/a, \\ \cosh n_3 y/a \sin n_4 y/a, \sinh n_3 y/a \sin n_4 y/a \}$$

Equating the coefficients of elements of matrix $[Y_R(y)]^T$ in equation (56) to zero, constants D_{1m} to D_{8m} are obtained in terms of C_{1m} to C_{8m} as

$$D_{1m} = K P_9 C_{1m} + D_{2m} = K P_9 C_{2m}$$

$$D_{3m} = K P_{10} C_{3m} + D_{4m} = K P_{10} C_{4m}$$

$$D_{4m} = K(P_7 C_{5m} + P_8 C_{6m}), \quad D_{6m} = K(P_7 C_{6m} + P_8 C_{7m})$$

$$D_{7m} = K(P_5 C_{6m} + P_6 C_{7m}), \quad D_{8m} = K(P_5 C_{5m} + P_6 C_{6m})$$

where

$$P_1 = - R_1/R_2 - \pi^2 a^2 + n_3^2 - n_4^2$$

$$P_2 = 2 n_3 n_4$$

$$P_3 = \pi^4 a^4 + n_3^4 + n_4^4 - 6n_3^2 - \pi^2 a^2 (n_3^2 - n_4^2)$$

$$P_4 = 4n_3^3 n_4 - 4n_4^3 n_3 - 2\pi^2 a^2 n_3 n_4$$

$$P_5 = (P_4 P_1 + P_3 P_2) / P$$

$$P_6 = (P_2 P_4 + P_1 P_3) / P$$

$$P_7 = (P_1 P_3 + P_2 P_4) / P$$

$$P_8 = (P_2 P_3 - P_1 P_4) / P$$

$$P_9 = (-R_1 / R_2) m^2 n^2 + n_1^2) / (m^4 n^4 + n_1^4 - m^2 n^2 n_1^2)$$

$$P_{10} = (-R_1 / R_2) m^2 n^2 + n_2^2) / (m^4 n^4 + n_2^4 - m^2 n^2 n_2^2)$$

$$K = -\frac{Eh a^2}{R_1} \quad P = P_3^2 + P_4^2$$

Matrix [A] has different values for different types of boundary conditions.

Simply supported along $y = 0$ and $y = b$

$$A_{11} = A_{13} = A_{15} = 1, \quad A_{12} = A_{14} = A_{16} = A_{17} = A_{18} = 0$$

$$A_{21} = \cosh n_1 b/a \quad A_{22} = \sinh n_1 b/a$$

$$A_{23} = \cosh n_2 b/a \quad A_{24} = \sinh n_2 b/a$$

$$A_{25} = \cosh n_3 b/a \cos n_4 b/a \quad A_{26} = \sinh n_3 b/a \cos n_4 b/a$$

$$A_{27} = \cos n_3 b/a \sin n_4 b/a \quad A_{28} = \sinh n_3 b/a \sin n_4 b/a$$

$$A_{31} = n_1^2 \quad A_{33} = n_2^2$$

$$A_{35} = n_3^2 - n_4^2 \quad A_{36} = 2n_3 n_4$$

$$A_{32} = A_{34} = 0 \quad A_{37} = A_{38} = 0$$

$$A_{41} = n_1^2 A_{21} \quad A_{42} = n_1^2 A_{22}$$

$$A_{43} = n_2^2 A_{23} \quad A_{44} = n_2^2 A_{24}$$

$$\begin{aligned}
 A_{46} &= A_{35} A_{26} - P_2 A_{27} & I_{45} &= I_{35} A_{25} - P_2 A_{28} \\
 I_{47} &= I_{35} + P_2 A_{26} & I_{48} &= I_{35} A_{28} + P_2 A_{25} \\
 A_{51} &= P_9 & I_{53} &= P_{10} \\
 A_{55} &= P_7 & I_{50} &= P_8 \\
 A_{52} &= A_{54} = 0 & A_{56} &= A_{57} = 0 \\
 I_{61} &= P_9 A_{21} & A_{62} &= P_9 A_{22} \\
 A_{63} &= P_{10} A_{23} & I_{64} &= P_{10} A_{24} \\
 I_{65} &= P_7 A_{25} + P_5 A_{28} & A_{66} &= P_7 A_{26} + P_5 A_{27} \\
 A_{67} &= P_8 A_{26} + P_6 A_{27} & I_{68} &= P_8 A_{25} + P_6 A_{28} \\
 A_{71} &= P_9 n_1^2 & A_{73} &= P_{10} n_2^2 \\
 A_{75} &= P_7 A_{35} + P_2 P_5 & A_{78} &= P_8 A_{35} + P_2 P_6 \\
 A_{72} &= A_{74} = 0 & I_{76} &= A_{77} = 0 \\
 I_{81} &= P_9 A_{41} & A_{82} &= P_9 A_{42} \\
 I_{83} &= P_{10} A_{43} & A_{84} &= P_{10} A_{44} \\
 A_{85} &= P_7 A_{45} + P_5 A_{48} & I_{86} &= P_7 A_{46} + P_5 A_{46} \\
 I_{87} &= P_8 A_{46} + P_6 A_{47} & A_{88} &= P_8 A_{45} + P_6 A_{48}
 \end{aligned}$$

Solution of the matrix equation (60), by putting the values of matrix $[I_{pm}]$ and matrix $[A]$, gives the values of constants, C_{1m} through C_{8m} . Deflection and moments may be obtained by substitution of C_{1m} through C_{8m} in Eqn. (61), (62, (63) and (64). Values of elements of the matrix $[A]$, for shallow shell with two edges simply supported and remaining two edges clamped, have been given in computer programme in Appendix C.

(d) Conclusion:

The method developed is more general than Ambartsuryan's⁽²⁾ method. It does not require to define deflection in term of any other functions such as $w = \nabla^4 \Psi_0$ where Ψ_0 is any arbitrary function. Stress function 'θ' and deflection function 'w' are replaced in term of Ψ_0 function in Eq. (29) in Ambartsuryan's method.

The solution of differential equation which is in term of Ψ_0 function, is obtained with the help of boundary conditions. Then stress function and deflection 'w' are obtained from Ψ_0 function. But the solution of differential equation which is in term of Ψ_0 function is mathematically difficult or most probably impossible for cases where $\nabla^2 \Psi_0$ is not zero. Method developed in preceding sections may be applied for cases where $\nabla^2 \Psi_0$ is not zero.

Present method can be used for any type of temperature distribution in their shallow shell. In such cases, the same method holds good except for addition of particular integral term in Eq. (50).

Parameter $a^2/h R_1$ and b/a ratio play an important role in magnitude and direction of deflection of thin shallow shell for fixed value of b/a ratio, dimensionless quantity for deflection of the shell decreases from upward direction to down-ward direction as values of the parameter $a^2/h R_1$ increases as shown in Appendix 'E', for all edges simply supported and

two edges simply supported and remaining two edges clamped. Similarly, for fixed value of parameter $a^2/h R_1$, dimensionless quantity for deflection of the shallow shell decreases from upward direction to down ward direction as values of the b/a ratio increases as shown in Appendix "A" for all edges simply supported and two edges simply supported and remaining two edges clamped. Values of dimensionless quantities for U_x and U_y decreases in magnitude with increasing the parameter $a^2/h R_1$ and b/a ratio. But change in magnitude due to parameter $a^2/h R_1$ and b/a ratio of dimensionless quantities of moments M_x and M_y is small in comparison to the change in magnitude and direction of dimensionless quantity for deflection. Magnitude and direction of the dimensionless quantity for moment M_{xy} also change with change of parameter $a^2/h R_1$ and b/a ratio as shown in Appendix B. For all cases numerical values and graphs have been presented for deflection and moments (U_x and U_y) at $x = a/2$ and $R_1/R_2 = 0.5$ for various values of parameter $a^2/h R_1$ and b/a ratio. This has been done for parameter $a^2/h R_1$ ranging from 5.0 to 20.0 at intervals of 5.0, and b/a ratio ranging from 1.0 to 2.0 at interval of 0.2. The deflection and moments have been presented by the dimensionless parameters $wh/\sigma E a^2$, U_x/Ea^2h^2 , U_y/Ea^2h^2 and M_{xy}/Ea^2h^2 . Dimensionless Parameter M_{xy}/Ea^2h^2 has been presented along $x = 0$. Only magnitude of dimensionless parameters U_x/Ea^2h^2 , U_y/Ea^2h^2 have been presented in Appendix B and direction is given in Table Appendix B. Since the variation of $wh/\sigma E a^2$, U_x/Ea^2h^2 , U_y/Ea^2h^2 along x -axis is in simple sine form, the values of these quantities at any point can be easily calculated.

The results obtained in preceding sections, may be used in design of shallow shell's roofs where outside the building, atmospheric temperature is different from inside temperature of the building. The method developed in preceding sections may be used in finding out the stresses in end caps of boilers and airwings where temperature gradient exists, due to temperature difference of inside and outside surfaces.

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APPENDIX - A

CHARACTERISTIC ROOTS OF THE DIFFERENTIAL EQUATION (47)

4.1 The homogeneous part of the differential equation (47-b) can be written as

$$A_{1m} \frac{d^6}{dy^6} Y_m(y) + A_{2m} \frac{d^5}{dy^5} Y_m(y) + A_{3m} \frac{d^4}{dy^4} Y_m(y) + A_{4m} \frac{d^3}{dy^3} Y_m(y) + A_{5m} Y_m(y) = 0 \quad (A-1)$$

where

A_{1m} to A_{5m} are given by eqn. (47-b)

The nature of the eight characteristic roots of Eq. (A-1) depends on the value of A_{1m} to A_{5m} and can be classified under the following cases.

Case 1 $\pm n_1, \pm n_2, \pm n_3, \pm n_4$

Case 2 $\pm n_1, \pm n_2, \pm n_3, \pm in_4$

Case 3 $\pm n_1, \pm n_2, \pm in_3, \pm in_4$

Case 4 $\pm n_1, \pm in_2, \pm in_3, \pm in_4$

Case 5 $\pm in_1, \pm in_2, \pm in_3, \pm in_4$

Case 6 $\pm n_1, \pm n_2, \pm (n_3 \pm in_4)$

Case 7 $\pm n_1, \pm in_2, \pm (n_3 \pm in_4)$

Case 8 $\pm in_1, \pm in_2, \pm (n_3 \pm in_4)$

Case 9 $\pm (n_1 \pm in_2), \pm (n_3 \pm in_4)$

where n_1, n_2, n_3 and n_4 are the components of the characteristic roots.

1.2 Basis of Solutions:

The basis of solutions $[Y_R(y)]^T$ (Eq. (51-c)) is given below as a row matrix $[Y_R(y)]^T$, along with the matrix $[N_R]$ (Eq. (52)), for various cases listed in Tens. (i-3)

Case 1

$$[Y_R(y)]^T = \{\cosh n_1 y, \sinh n_1 y, \cosh n_2 y, \sinh n_2 y, \cosh n_3 y, \sinh n_3 y, \cosh n_4 y, \sinh n_4 y\} \quad (A-3a)$$

$$[N_R] = \begin{bmatrix} 0 & d_1 n_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ n_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_2 n_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & n_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_3 n_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & n_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_4 n_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_4 & 0 \end{bmatrix} \quad (A-3b)$$

$$\text{where } d_1 = d_2 = d_3 = d_4 = 1$$

Case 2

$$[Y_R(y)]^T = \{\cosh n_1 y, \sinh n_1 y, \cosh n_2 y, \sinh n_2 y, \cosh n_3 y, \sinh n_3 y, \cos n_4 y, \sin n_4 y\} \quad (A-4a)$$

$[N_R]$ is given by Eq. (A-3b) with

$$d_1 = d_2 = d_3 = 1 \text{ and } d_4 = -1 \quad (A-4b)$$

Case 3

$$[Y_R(y)]^T = \{\cosh n_1 y, \sinh n_1 y, \cosh n_2 y, \sinh n_2 y, \cos n_3 y, \sin n_3 y, \cos n_4 y, \sin n_4 y\} \quad (A-5a)$$

$[Y_R]$ is given by Eq. (A-3b) with

$$d_1 = d_2 = 1, \text{ and } d_3 = d_4 = -1 \quad (A-5b)$$

Case 4

$$[Y_R(y)]^T = \{\cosh n_1 y, \sinh n_1 y, \cos n_2 y, \sin n_2 y, \cos n_3 y, \sin n_3 y, \cos n_4 y, \sin n_4 y\} \quad (A-6a)$$

$[Y_R]$ is given by Eq. (A-3b) with

$$d_1 = 1, \text{ and } d_2 = d_3 = d_4 = -1 \quad (A-6b)$$

Case 5

$$[Y_R(y)]^T = \{\cos n_1 y, \sin n_1 y, \cos n_2 y, \sin n_2 y, \cos n_3 y, \sin n_3 y, \cos n_4 y, \sin n_4 y\} \quad (A-7a)$$

$[Y_R]$ is given by Eq. (A-3b) with

$$d_1 = d_2 = d_3 = d_4 = -1 \quad (A-7b)$$

Case 6

$$[Y_R(y)]^T = \{\cosh n_1 y, \sinh n_1 y, \cosh n_2 y, \sinh n_2 y, \cosh n_3 y, \cos n_3 y, \sinh n_3 y, \cos n_4 y, \sinh n_3 y, \sin n_4 y, \cosh n_3 y, \sin n_4 y\} \quad (A-8a)$$

$$[X_R] = \begin{bmatrix} 0 & d_1 n_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ n_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_2 n_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & n_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_3 & -n_4 \\ 0 & 0 & 0 & 0 & 0 & n_3 & 0 & -n_4 \\ 0 & 0 & 0 & 0 & n_3 & 0 & -n_4 & 0 \\ 0 & 0 & 0 & 0 & n_4 & 0 & n_3 & 0 \end{bmatrix} \quad (A-8b)$$

where

$$d_1 = d_2 = 1$$

Case 7

$$[Y_R(y)]^T = \{ \cosh n_1 y, \sinh n_1 y, \cos n_2 y, \sin n_2 y, \cosh n_3 y, \cos n_4 y, \sinh n_3 y, \cos n_4 y, \sin n_3 y, \sin n_4 y, \cosh n_3 y, \sin n_4 y \} \quad (A-9a)$$

$[X_R]$ is given by Eq. (A-8b) with

$$d_1 = 1, \text{ and } d_2 = -1 \quad (A-9b)$$

Case 8

$$[Y_R(y)]^T = \{ \cos n_1 y, \sin n_1 y, \cos n_2 y, \sin n_2 y, \cosh n_3 y, \cos n_4 y, \sin n_3 y, \cos n_4 y, \sinh n_3 y, \sin n_4 y, \cosh n_3 y, \sin n_4 y \} \quad (A-10a)$$

$[X_R]$ is given by Eq. (A-8b) with

$$d_1 = d_2 = -1 \quad (A-10b)$$

Case 9

$$[\gamma_R(y)]^T = \left\{ \begin{array}{cccc} \cosh a_1 y & \cos a_2 y, & \sinh a_1 y & \cos a_2 y, \\ \cosh a_1 y & \sin a_2 y, & \sinh a_1 y & \sin a_2 y, \\ \cosh a_3 y & \cos a_4 y, & \sinh a_3 y & \cos a_4 y, \\ \cosh a_3 y & \sin a_4 y, & \sinh a_3 y & \sin a_4 y \end{array} \right\}$$

(A-11a)

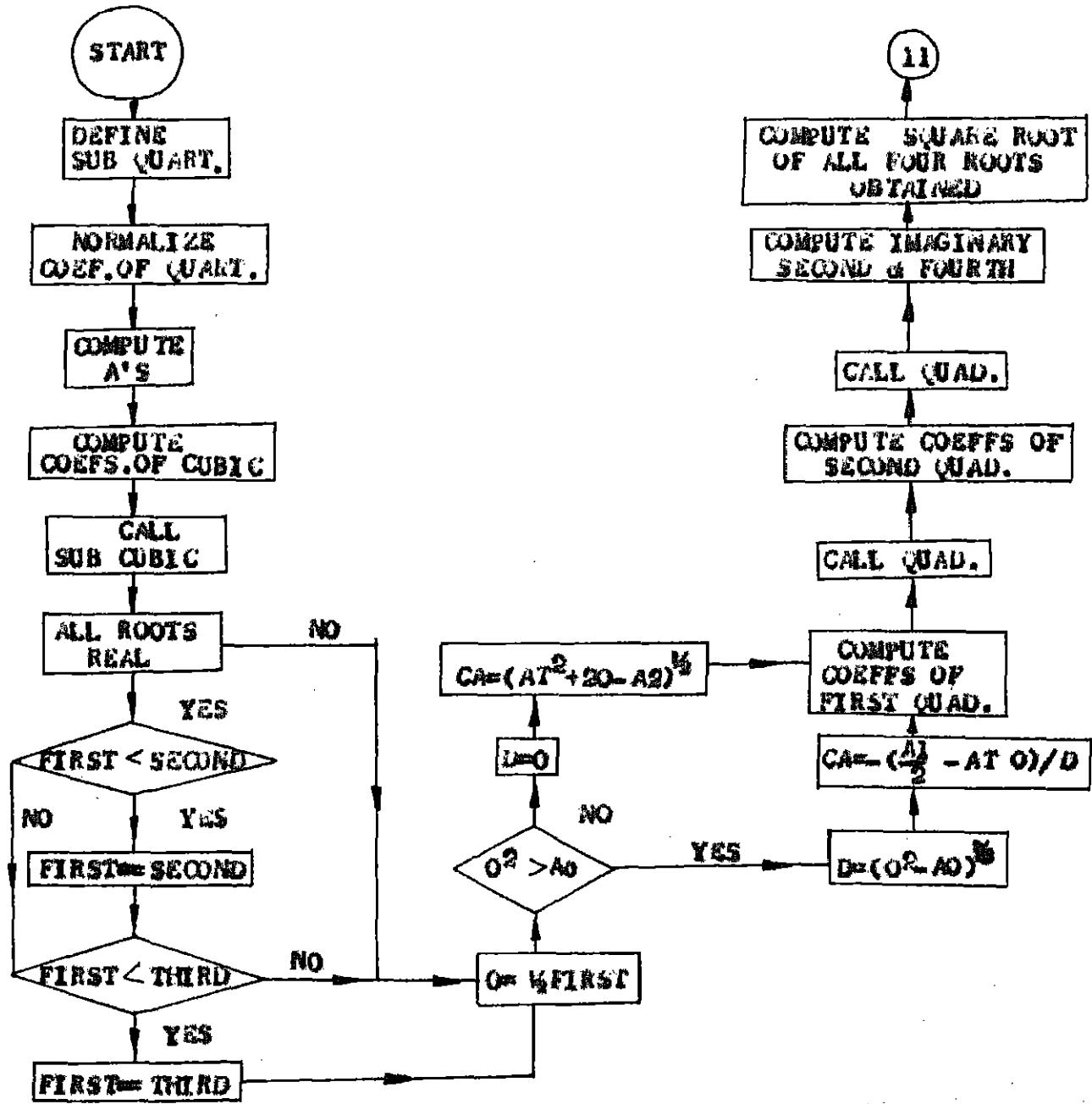
$$[N_R] = \begin{bmatrix} 0 & a_1 & 0 & -a_2 & 0 & 0 & 0 & 0 \\ a_1 & 0 & -a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & a_1 & 0 & 0 & 0 & 0 \\ a_2 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_3 & 0 & -a_4 & 0 \\ 0 & 0 & 0 & 0 & a_3 & 0 & -a_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_4 & 0 & a_3 \\ 0 & 0 & 0 & 0 & a_4 & 0 & a_3 & 0 \end{bmatrix}$$

(A-11b)

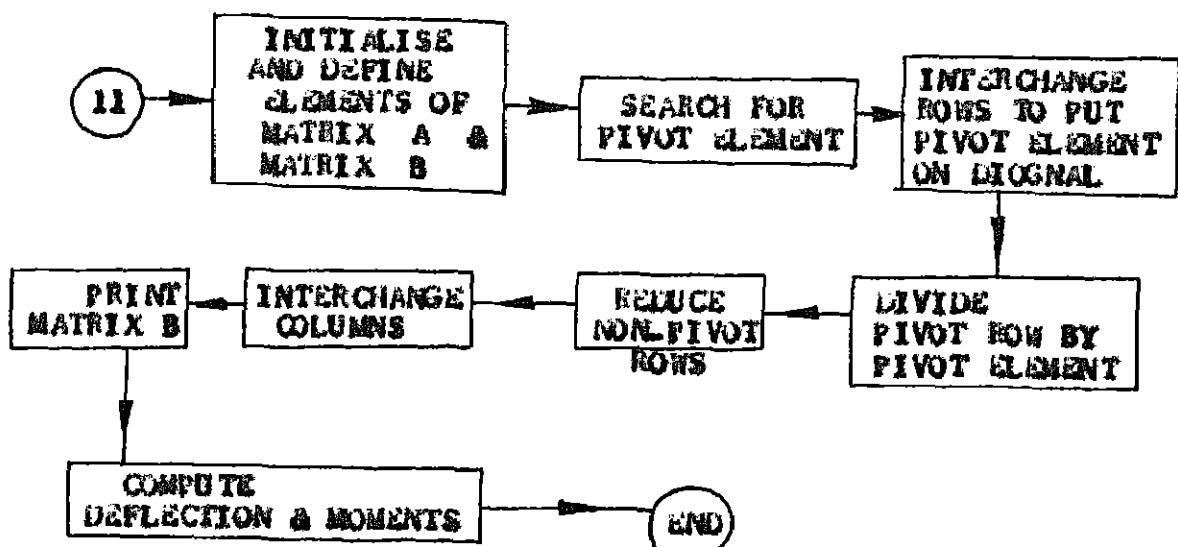
APPENDIX B

FLOW CHARTS

Main Flow Chart



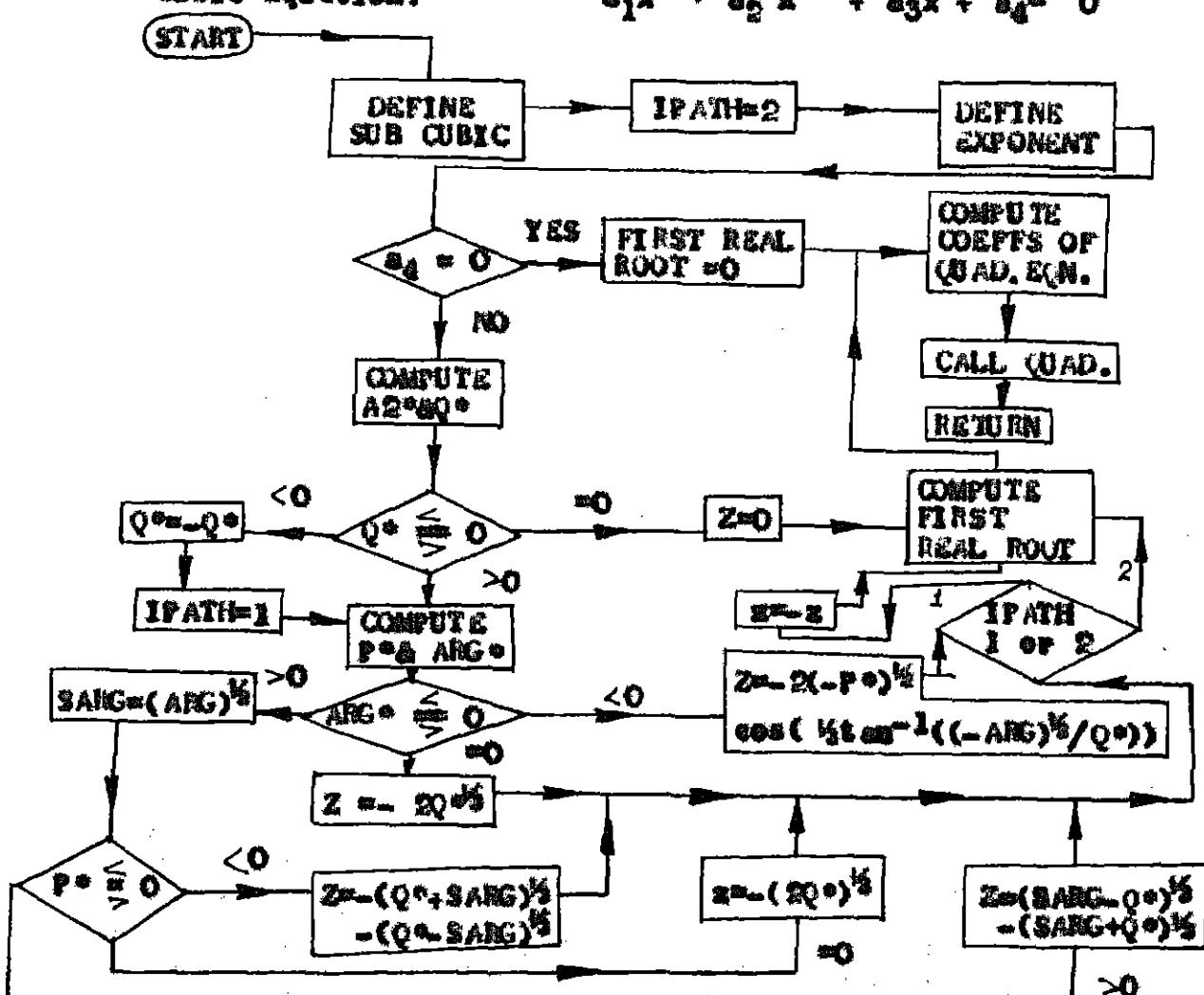
*Expressions are given in computer programme.



Sub Flow Charts

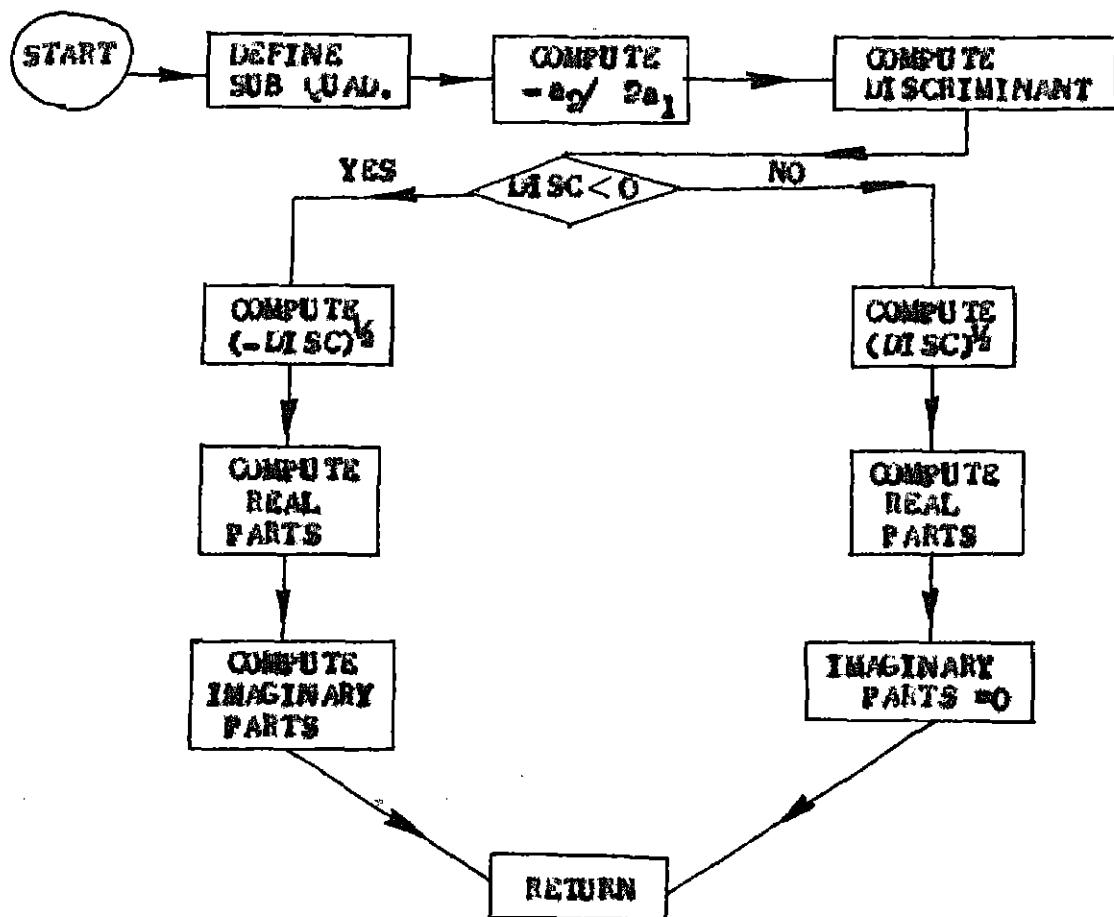
Cubic Equation:

$$a_1x^3 + a_2x^2 + a_3x + a_4 = 0$$



Quadratic Equation:

$$a_1x^2 + a_2x + a_3 = 0$$



APPENDIX C

ALL EDGES SIMPLY SUPPORTED FORTRAN SOURCE LIST
SOURCE STATEMENT

FORTRAN SOURCE LIST

SOURCE STATEMENT

```

7 PRINT7,PF
FORMAT(1X,F7.2)
CC1 I=1,4
RR(I)=X(I)/XR(I)
THETA(I)=ATAN(RR(I))
PI=4.*ATAN(1.)
RP(I)=SQRT(XR(I)**2+XI(I)**2)
RR(I)=SQRT(RP(I))
ER(I,1)=RR(I)*COS(PI+0.5*THETA(I))
EI(I,2)=RR(I)*SIN(2.*PI+0.5*THETA(I))
ER(I,2)=RR(I)*COS(2.*PI+0.5*THETA(I))
PRINTC,((ER(I,J),J=1,4),J=1,2)
PRINT8,((EI(I,J),I=1,4),J=1,2)
FORMAT(1X,PE14.4)
INITIALIZATION
Q(1)=ER(1,2)
Q(2)=ER(2,2)
Q(3)=ER(4,2)
Q(4)=EI(4,2)
G=3.1416
P(1)=-R *(PM*G)**2+ Q(3)**2- Q(4)**2
P(2)=2.*Q(3)*Q(4)
P(3)=(PM*G)**4+Q(3)**4+Q(4)**4-6.* (Q(3)*Q(4))**2-(Q(3)**2-Q(4)**2)
2*(PM*G)**2
P(4)=4.*Q(3)**3*Q(4)-4.*Q(4)**3*Q(3)-2.*Q(4)*Q(3)*(PM*G)**2
PK=P(3)**2+P(4)**2
P(5)=(P(4)*P(1)-P(3)*P(2))/PK
P(6)=(P(2)*P(4)+P(1)*P(3))/PK
P(7)=(P(1)*P(3)+P(2)*P(4))/PK
P(8)=(P(2)*P(3)-P(1)*P(4))/PK
P(9)=(-R *(PM*G)**2+Q(1)**2)/((PM*G)**4+Q(1)**4-(PM*G*Q(1))**2)
P(10)=(-R *(PM*G)**2+Q(2)**2)/((PM*G)**4+Q(2)**4-(PM*G*Q(2))**2)
A(1,1)=1.0
A(1,2)=0.0
A(1,3)=1.0
A(1,4)=0.0
A(1,5)=1.0
A(1,6)=0.0
A(1,7)=0.0
A(1,8)=0.0
A(2,1)=COSH(Q(1)*F)
A(2,2)=SINH (Q(1)*F)
A(2,3)=COSH(Q(2)*F)
A(2,4)=SINH (Q(2)*F)
A(2,5)=COSH(Q(3)*F)*COS(Q(4)*F)
A(2,6)=SINH (Q(3)*F)*COS(Q(4)*F)
A(2,7)=COSH(Q(3)*F)*SIN(Q(4)*F)
A(2,8)=SINH (Q(3)*F)*SIN(Q(4)*F)
A(3,1)=Q(1)**2
A(3,2)=0.0
A(3,3)=Q(2)**2
A(3,4)=0.0
A(3,5)=Q(3)**2-Q(4)**2
A(3,6)=0.0
A(3,7)=0.0

```

FORTRAN SOURCE LIST

SOURCE STATEMENT

```

A(3,8)=2.*C(3)*C(4)
A(4,1)=C(1)**2*A(2,1)
A(4,2)=C(1)**2*A(2,2)
A(4,3)=C(2)**2*A(2,3)
A(4,4)=C(2)**2*A(2,4)
A(4,5)=A(3,5)*A(2,5)-P(2)*A(2,8)
A(4,6)=A(3,5)*A(2,6)-P(2)*A(2,7)
A(4,7)=(A(3,5)*A(2,7)+P(2)*A(2,6))
A(4,8)=(A(3,5)*A(2,8)+P(2)*A(2,5))
A(5,1)=P(1)
A(5,2)=0.
A(5,3)=P(10)
A(5,4)=0.
A(5,5)=P(7)
A(5,6)=0.
A(5,7)=0.
A(5,8)=P(8)
A(6,1)=P(9)*A(2,1)
A(6,2)=P(9)*A(2,2)
A(6,3)=P(10)*A(2,3)
A(6,4)=P(10)*A(2,4)
A(6,5)=P(7)*A(2,5)+P(5)*A(2,8)
A(6,6)=P(7)*A(2,6)+P(5)*A(2,7)
A(6,7)=P(8)*A(2,6)+P(6)*A(2,7)
A(6,8)=P(8)*A(2,5)+P(6)*A(2,8)
A(7,1)=C(1)**2*P(9)
A(7,2)=0.
A(7,3)=C(2)**2*P(10)
A(7,4)=0.
A(7,5)=P(7)*A(3,5)+P(2)*P(5)
A(7,6)=0.
A(7,7)=0.
A(7,8)=P(8)*A(3,5)+P(2)*P(6)
A(8,1)=A(4,1)*P(9)
A(8,2)=A(4,2)*P(9)
A(8,3)=A(4,3)*P(10)
A(8,4)=A(4,4)*P(10)
A(8,5)=P(7)*A(4,5)+P(5)*A(4,8)
A(8,6)=P(7)*A(4,6)+P(5)*A(4,7)
A(8,7)=P(8)*A(4,6)+P(6)*A(4,7)
A(8,8)=P(8)*A(4,5)+P(6)*A(4,8)
B(1,1)=-1.0
B(2,1)=-1.0
B(3,1)=-(PM*G)**2
B(4,1)=-(PM*G)**2
B(5,1)=0.
B(6,1)=0.
B(7,1)=0.
B(8,1)=0.
N=8
M=1
DETERM=0.0
15 DO 21 J=1,N
21 IPIVOT(J)=0
30 DO 550 I=1,N

```

SOURCE STATEMENT

```

C      SEARCH FOR PIVOT ELEMENT
C
45  AMAX=0.
46  DO 105 J=1,N
50  IF(IPIVCT(J)-1)60,105,60
60  DO 100 K=1,N
70  IF(IPIVCT(K)-1)80,100,740
80  IF(ABS(AMAX)-ABS(A(J,K)))85,100,100
85  IROW=J
90  ICOLUMN=K
95  AMAX=A(J,K)
100 CONTINUE
105 CONTINUE
C      INTERCHANGE ROWS TO PUT PIVOT ELEMENT ON DIAGONAL
C
110 IPIVCT(ICOLUMN)=IPIVCT(ICOLUMN)+1
130 IF(IROW-ICOLUMN)140,260,140
140 DETERM=-DETERM
150 DO 200 L=1,N
160 SWAP=A(IROW,L)
170 A(IROW,L)=A(ICOLUMN,L)
180 A(ICOLUMN,L)=SWAP
205 IF(M)260,260,210
210 DO 250 L=1,M
220 SWAP=B(IROW,L)
230 B(IROW,L)=B(ICOLUMN,L)
250 B(ICOLUMN,L)=SWAP
260 INDEX(I,1)=IROW
270 INDEX(I,2)=ICOLUMN
310 PIVOT(I)=A(ICOLUMN,ICOLUMN)
320 DETERM=DETERM*PIVOT(I)
C      DIVIDE PIVOT ROW BY PIVOT ELEMENT
C
330 A(ICOLUMN,ICOLUMN)=1.
340 DO 350 L=1,N
350 A(ICOLUMN,L)=A(ICOLUMN,L)/PIVOT(I)
355 IF(M)380,380,360
360 DO 370 L=1,M
370 B(ICOLUMN,L)=B(ICOLUMN,L)/PIVOT(I)
C      REDUCE NON-PIVOT ROWS
C
380 DO 550 L1=1,N
390 IF(L1-ICOLUMN)400,550,400
400 TA=A(L1,ICOLUMN)
420 A(L1,ICOLUMN)=0.
430 DO 450 L=1,N
450 A(L1,L)=A(L1,L)-A(ICOLUMN,L)*TA
455 IF(M)550,550,460
460 DO 500 L=1,M
500 B(L1,L)=B(L1,L)-B(ICOLUMN,L)*TA
550 CONTINUE

```

SOURCE STATEMENT

INTERCHANGE COLUMNS

```

600 DO 710 I=1,N
610 L=N+1-I
620 IF( INDEX(L,1)-INDEX(L,2) )630,710,630
630 JROW=INDEX(L,1)
640 JCOLUMN=INDEX(L,2)
650 GO TO 705 K=1,N
660 SWAP=A(K,JROW)
670 A(K,JROW)=A(K,JCOLUMN)
700 A(K,JCOLUMN)=SWAP
705 CONTINUE
710 CONTINUE
740 CONTINUE
    PRINT#20,(B(I,1),I=1,8)
820 FORMAT(1X,8E15.4)
Y=0.0
1900 T(1)=COSH(Q(3)*F*Y)
T(2)=SINH (Q(3)*F*Y)
T(3)=COS (Q(4)*F*Y)
T(4)=SIN(Q(4)*F*Y)
T(5)=COSH(Q(1)*F*Y)
T(6)=SINH (Q(1)*F*Y)
T(7)=COSH(Q(2)*F*Y)
T(8)=SINH (Q(2)*F*Y)
W(1)=B(1,1)*T(5)
W(2)=B(2,1)*T(6)
W(3)=B(3,1)*T(7)
W(4)=B(4,1)*T(8)
W(5)=B(5,1)*T(1)*T(3)
W(6)=B(6,1)*T(2)*T(3)
W(7)=B(7,1)*T(1)*T(4)
W(8)=B(8,1)*T(2)*T(4)
W(9)=W(1)+W(2)+W(3)+W(4)+W(5)+W(6)+W(7)+W(8)+1.0
WK=5.2*W(9)/(PM*G)**3
XM(1)=-W(9)
XM(2)=Q(1)**2*(B(1,1)*T(5)+B(2,1)*T(6))
XM(3)=Q(2)**2*(B(3,1)*T(7)+B(4,1)*T(8))
WK=W(5)+W(6)+W(7)+W(8)
XM(4)=(Q(3)**2-Q(4)**2)*WK
XM(5)=2.*Q(3)*Q(4)*(-B(5,1)*T(2)*T(4)-B(6,1)*T(1)*T(4)+B(7,1)*T(2)
2*T(3)+B(8,1)*T(1)*T(3))
ZM=(XM(2)+XM(3)+XM(4)+XM(5))/(PM*G)**2
XMM=- (XM(1)+.3*ZM+1.0)/(2.1*PM*G)
YMM=- (ZM+.3*XMM(1)+1.)/(2.1*PM*G)
XYM(1)=Q(1)*(B(1,1)*T(6)+B(2,1)*T(5))
XYM(2)=Q(2)*(B(3,1)*T(8)+B(4,1)*T(7))
XYM(3)=Q(3)*(B(5,1)*T(2)*T(3)+B(6,1)*T(1)*T(3))
XYM(4)=Q(3)*(-B(7,1)*T(2)*T(4)+B(8,1)*T(1)*T(4))
XYM(5)=Q(4)*(-B(5,1)*T(1)*T(4)-B(6,1)*T(2)*T(4))
XYM(6)=Q(4)*(B(7,1)*T(1)*T(3)+B(8,1)*T(2)*T(3))
XYM(7)=XYM(1)+XYM(2)+XYM(3)+XYM(4)+XYM(5)+XYM(6)
XKM=-XYM(7)/(3.0*(PM*G)**2)

```

FORTRAN SOURCE LIST

SOURCE STATEMENT

```
2      PRINT2000,Y,WW,XNN,YNN,XNN
3 2000 FORMAT(1X,5E15.4)
4      Y=Y+.1
5      IF(Y-1.0)1900,1900,2100
6 2100 F=F+.2
7      IF(F-2.)178,78,2200
8 2200 X=X+5.0
9      IF(X-50.0)12,2,6
10 6      PM=PM+2.0
11 3      IF(PM-7.)4,4,9
12 9      STOP
13 5      END
```

IBMAP ASSEMBLY

MESSAGES FOR ABOVE ASSEMBLY

N SOURCE STATEMENT

```

0 SUBFTC CUBIC  NOCHECK
1      SUBROUTINE CUBIC(A,XR,XI)
2      DIMENSION A(4),XR(3),AQ(3)
3      IPATH=2
4      FX=1./3.
5      1050 FORMAT(5E14.2)
6      IF(A(4))1006,1004,1006
7      1004 XR(1)=0.
8      GO TO 1034
9      1006 A2=A(1)**2
0      Q=(27.*A2*A(4)-9.*A(1)*A(2)*A(3)+2.*A(2)**3)/(54.*A2*A(1))
1      IF(Q)1010,1008,1014
2      1007 Z=0.
3      GO TO 1032
4      1010 C=-Q
5      IPATH=1
6      1014 P=(3.*A(1)*A(3)-A(2)**2)/(9.*A2)
7      ARG=P**3+Q**2
8      IF(ARG)1016,1018,1020
9      1016 Z=-2.*SQRT(-P)*COS(ATAN(SQRT(-ARG)/Q)/3.)
0      GO TO 1028
1      1018 Z=-2.*Q**EX
2      GO TO 1028
3      1020 SARG=SQRT(ARG)
4      IF(P)1022,1024,1026
5      1022 Z=-(C+SARG)**EX-(C-SARG)**EX
6      GO TO 1028
7      1024 Z=-(2.*Q)**FX
8      GO TO 1028
9      1026 Z=(SARG-Q)**EX-(SARG+C)**EX
0      1028 GO TO(1030,1032),IPATH
1      1030 Z=-Z
2      1032 XR(1)=(3.*A(1)*Z-A(2))/(3.*A(1))
3      1034 AQ(1)=A(1)
4      AQ(2)=A(2)+XR(1)*A(1)
5      AQ(3)=A(3)+XR(1)*AQ(2)
6      CALL QUAD(AQ,XR(2),XR(3),XI)
7      RETURN
8      END

```

IBMAP ASSEMBLY CUBIC

MESSAGES FOR ABOVE ASSEMBLY

FORTRAN SOURCE LIST

SOURCE STATEMENT

```
0 $IBFTC QUAD      NODECK
1      SUBROUTINE QUAD(A,XR1,XR2,XI)
2      DIMENSION A(3)
3      26 FORMAT(5F14.3)
4      X1=-A(2)/(2.*A(1))
5      DISC=X1**2-A(3)/A(1)
6      IF(DISC)27,28,28
7      27 X2=SQRT(-DISC)
8      XR1=X1
9      XR2=X1
10     XI=X2
11     GO TO 29
12     28 X2=SQRT(DISC)
13     XR1=X1+X2
14     XR2=X1-X2
15     XI=0.
16     29 RETURN
17     END
```

IBMAP ASSEMBLY QUAD

MESSAGES FOR ABOVE ASSEMBLY

IBLDR -- JOB 000000

IS BEING ENTERED INTO STORAGE.

EIGENVALUE SOURCE LIST

SOURCE STATEMENT

```

0 LIBRARY 100DECK
1      DIMENSION C(5),XT(4),AC(4),BC(4),XI(4),AD(4),XP(4)
2      DIMENSION THETA(4),RP(4),PR(4),F(4,2),FI(4,1),PF(4)
3      DIMENSION IPIVOT(8),A(6,2),B(6,1),INDEX(6,2),PIVOT(8)
4      DIMENSION P(15),R(9),L(3),XK(5),Q(4),XK(7)

C
5      F=0/A
6      PR=J*F
7      PRINT2
8      FORMAT(1H1)
9      R=0.0
10     Y=0.0
11     F=1.0
12     PII=3.1416
13     FACTR=PII*PR
14     V=0.3
15     U=12.*{1.0-V**2}*X**2
16     C(1)=1.0
17     C(2)=-4.0*(FACTR)**2
18     C(3)=6.0*(FACTR)**4+U
19     C(4)=-2.*(FACTR)**2*{(2.*FACTR)**4+U*R}
20     C(5)=(FACTR)**4*{(FACTR)**4+R**2*U}
21     A3=C(2)/C(1)
22     A2=C(3)/C(1)
23     A1=C(4)/C(1)
24     AD=C(5)/C(1)
25     AT=A3/2.
26     AC(1)=1.
27     AC(2)=-A2
28     AC(3)=A1*A3-4.*AD
29     AC(4)=AD*(4.*A2-A3**2)-A1**2
30     CALL CUBIC(AC,RT,RTI)
31     IF(RTI)20,10,20
32     10 IF(IRT(1)=RT(2))11,12,12
33     11 RT(1)=RT(2)
34     12 IF(IRT(1)=RT(3))13,20,20
35     13 RT(1)=RT(3)
36     20 Q=RT(1)/2.
37     21 IF(Q**2-AD)22,22,24
38     22 D=0.
39     23 CA=SQRT(AT**2+2.*D-A2)
40     24 D=SQRT(D**2-AD)
41     25 CA=-(A1/2.-AT*Q)/D
42     26 AQ(1)=1.
43     27 AQ(2)=AT-CA
44     28 AQ(3)=Q-D
45     29 CALL QUAD(AQ,XR(1),XR(2),XI(1))
46     30 Q(1)=1.
47     31 Q(2)=AT+CA
48     32 Q(3)=D+Q
49     33 CALL QUAD(BQ,XR(3),XR(4),XI(3))
50     34 XI(2)=-XI(1)
51     35 XI(4)=-XI(3)
52     PRINT77,X
53     FORMAT(1X,E7.2)
54

```

FORTRAN SOURCE LIST

SOURCE STATEMENT

```

6      PRINTF,PF
7      FORMAT(1X,F7.2)
8      DO 1 J=1,4
1      RR(I)=XI(I)/XR(I)
2      THETA(I)=ATAN(RR(I))
3      PI=4.*ATAN(1.)
4      RQ(I)=SQRT(XR(I)**2+XI(I)**2)
5      RQ(I)=SQRT(RQ(I))
6      FR(I,1)=RR(I)*COS(PI+0.5*THETA(I))
7      FI(I,1)=RR(I)*SIN(PI+0.5*THETA(I))
8      FI(I,2)=RR(I)*SIN(2.*PI+0.5*THETA(I))
1      FR(I,2)=RR(I)*COS(2.*PI+0.5*THETA(I))
3      PRINTF,((FR(I,J)),I=1,6),J=1,2)
4      PRINTF,((FI(I,J)),I=1,4),J=1,2)
5      ENDKAT(1X,8E14.4)
C
C. INITIALIZATION
6      Q(1)=FR(1,2)
7      Q(2)=FR(2,2)
8      Q(3)=FR(4,2)
1      Q(4)=FI(4,2)
2      G=3.1416
3      P(1)=-.5*(PM*G)**2+ Q(3)**2- Q(4)**2
4      P(2)=2.*Q(3)*Q(4)
5      P(3)=(PM*G)**4+Q(3)**4+Q(4)**4-6.*Q(3)*Q(4))**2-(Q(3)**2-Q(4)**2)
2*(PM*G)**2
6      P(4)=4.*Q(3)**3*Q(4)-4.*Q(4)**3*Q(3)-2.*G(4)*Q(3)*(PM*G)**2
7      PK=P(3)**2+P(4)**2
8      P(5)=(P(4)*P(1)-P(3)*P(2))/PK
1      P(6)=(P(2)*P(4)+P(1)*P(3))/PK
2      P(7)=(P(1)*P(3)+P(2)*P(4))/PK
3      P(8)=(P(2)*P(3)-P(1)*P(4))/PK
4      P(9)=(-.5*(PM*G)**2+Q(1)**2)/((PM*G)**4+Q(1)**4-(PM*G*Q(1)**2))
5      P(10)=(-.5*(PM*G)**2+Q(2)**2)/((PM*G)**4+Q(2)**4-(PM*G*Q(2)**2))
6      A(1,1)=1.0
7      A(1,2)=0.0
8      A(1,3)=1.0
1      A(1,4)=0.0
2      A(1,5)=1.0
3      A(1,6)=0.0
4      A(1,7)=0.0
5      A(1,8)=0.0
6      A(2,1)=COSH(Q(1)*F)
7      A(2,2)=SINH(Q(1)*F)
8      A(2,3)=COSH(Q(2)*F)
1      A(2,4)=SINH(Q(2)*F)
2      A(2,5)=COSH(Q(3)*F)*COS(Q(4)*F)
3      A(2,6)=SINH(Q(3)*F)*COS(Q(4)*F)
4      A(2,7)=COSH(Q(3)*F)*SIN(Q(4)*F)
5      A(2,8)=SINH(Q(3)*F)*SIN(Q(4)*F)
6      A(3,1)=0.0
7      A(3,2)=Q(1)
8      A(3,3)=0.0
1      A(3,4)=Q(2)
2      A(3,5)=0.0
3      A(3,6)=Q(3)

```

FORTRAN SOURCE LIST

SUBL-CC STATEMENT

```

4 A(3,7)=Q(4)
5 A(3,8)=0.0
6 A(4,1)=Q(1)*A(2,2)
7 A(4,2)=Q(1)*A(2,1)
8 A(4,3)=Q(2)*A(2,4)
9 A(4,4)=Q(2)*A(2,3)
10 A(4,5)=Q(3)*A(2,6)-Q(4)*A(2,7)
11 A(4,6)=Q(3)*A(2,5)-Q(4)*A(2,4)
12 A(4,7)=Q(3)*A(2,4)+Q(4)*A(2,5)
13 A(4,8)=Q(3)*A(2,7)+Q(4)*A(2,6)
14 A(5,1)=P(9)
15 A(5,2)=0.0
16 A(5,3)=P(10)
17 A(5,4)=0.0
18 A(5,5)=P(7)
19 A(5,6)=0.0
20 A(5,7)=0.0
21 A(5,8)=P(8)
22 A(6,1)=P(9)*A(2,1)
23 A(6,2)=P(9)*A(2,2)
24 A(6,3)=P(10)*A(2,3)
25 A(6,4)=P(10)*A(2,4)
26 A(6,5)=P(7)*A(2,5)+P(5)*A(2,8)
27 A(6,6)=P(7)*A(2,6)+P(5)*A(2,7)
28 A(6,7)=P(8)*A(2,6)+P(6)*A(2,7)
29 A(6,8)=P(8)*A(2,5)+P(6)*A(2,8)
30 P(11)=Q(3)**2-Q(4)**2
31 P(12)=Q(1)**2*A(2,1)
32 P(13)=Q(1)**2*A(2,2)
33 P(14)=Q(2)**2*A(2,3)
34 P(15)=Q(2)**2*A(2,4)
35 P(16)=P(11)*A(2,5)-P(2)*A(2,8)
36 P(17)=P(11)*A(2,6)-P(2)*A(2,7)
37 P(18)=P(11)*A(2,7)+P(2)*A(2,6)
38 P(19)=P(11)*A(2,8)+P(2)*A(2,5)
39 A(7,1)=Q(1)**2*P(9)
40 A(7,2)=0.0
41 A(7,3)=Q(2)**2*P(10)
42 A(7,4)=0.0
43 A(7,5)=P(7)*P(11)+P(2)*P(5)
44 A(7,6)=0.0
45 A(7,7)=0.0
46 A(7,8)=P(8)*P(11)+P(2)*P(6)
47 A(8,1)=P(12)*P(9)
48 A(8,2)=P(13)*P(9)
49 A(8,3)=P(14)*P(10)
50 A(8,4)=P(15)*P(10)
51 A(8,5)=P(7)*P(16)+P(5)*P(19)
52 A(8,6)=P(7)*P(17)+P(5)*P(18)
53 A(8,7)=P(8)*P(17)+P(6)*P(18)
54 A(8,8)=P(8)*P(16)+P(6)*P(19)
55 Q(3,1)=-1.0
56 Q(2,1)=-1.0
57 B(3,1)=0.0
58 B(4,1)=0.0

```

FORTRAN SOURCE LIST

SOURCE STATEMENT

```

3  B(5,1)=0.0
4  B(6,1)=5.0
5  B(7,1)=1.0
6  B(8,1)=0.0
7  K=8
8  M=1
9  DETERM=0.0
10 14 DO J=1,M
11  IPIVOT(J)=0
12  30 DO 500 I=1,N

```

SEARCH FOR PIVOT ELEMENT

```

40  AMAX=0.
45  DO 140 J=1,N
50  IF(IPIVOT(J)-1)60,105,60
60  DO 140 K=1,N
70  IF(IPIVOT(K)-1)80,100,740
80  IF(ABS(AMAX)-ABS(A(J,K)))85,100,100
85  IROW=J
90  ICOLUMN=K
95  AMAX=A(J,K)
100 CONTINUE
105 CONTINUE

```

INTERCHANGE ROWS TO PUT PIVOT ELEMENT ON DIAGONAL

```

110 IPIVOT(ICOLUMN)=IPIVOT(ICOLUMN)+1
130 IF(IROW-ICOLUMN)140,260,140
140 DETERM=-DETERM
150 DO 260 L=1,N
160 SWAP=A(IROW,L)
170 A(IROW,L)=A(ICOLUMN,L)
180 A(ICOLUMN,L)=SWAP
190 IF(M)260,260,210
210 DO 250 L=1,M
220 SWAP=B(IROW,L)
230 B(IROW,L)=B(ICOLUMN,L)
240 B(ICOLUMN,L)=SWAP
250 INDEX(I,1)=IROW
260 INDEX(I,2)=ICOLUMN
270 PIVOT(I)=A(ICOLUMN,ICOLUMN)
280 DETERM=DETERM*PIVOT(I)

```

DIVIDE PIVOT ROW BY PIVOT ELEMENT

```

330 A(ICOLUMN,ICOLUMN)=1.
340 DO 350 L=1,N
350 A(ICOLUMN,L)=A(ICOLUMN,L)/PIVOT(I)
355 IF(M)360,380,360
360 DO 370 L=1,M
370 B(ICOLUMN,L)=B(ICOLUMN,L)/PIVOT(I)

```

REDUCE NON-PIVOT ROWS

FORTRAN SOURCE LIST

SUBJECT STATEMENT

```

35 300 DO 500 LI=1,N
36 300 IF(LI-ICOLUMN)400,550,400
37 400 TA=*(LI,ICOLUMN)
38 420 A(LI,ICOLUMN)=0.
39 430 DO 450 L=1,N
40 450 A(LI,L)=A(LI,L)-A(ICOLUMN,L)*TA
41 450 IF(0)550,530,460
42 460 DO 500 L=1,N
43 500 B(LI,L)=B(LI,L)-B(ICOLUMN,L)*TA
44 500 CONTINUE

```

INTERCHANGE COLUMNS

```

53 500 DO 710 I=1,N
54 510 L=N+1-I
55 520 IF(LINDEX(L,1)-INDEX(L,2))630,710,630
56 530 JROW=INDEX(L,1)
57 540 JCOLUMN=INDEX(L,2)
58 550 DO 705 K=1,N
59 560 SWAP=A(K,JROW)
60 570 A(K,JCOLUMN)=A(K,JROW)
61 580 A(K,JCOLUMN)=SWAP
62 590 CONTINUE
63 710 CONTINUE
64 740 CONTINUE
65 740 PRINT820,(B(I,1),I=1,8)
66 820 FORMAT(1X,8E15.4)
67 820 Y=0.0
68 1900 T(1)=COSH(Q(3)*F*Y)
69 1900 T(2)=SINH (Q(3)*F*Y)
70 1900 T(3)=COS (Q(4)*F*Y)
71 1900 T(4)=SIN(Q(4)*F*Y)
72 1900 T(5)=COSH(Q(1)*F*Y)
73 1900 T(6)=SINH (Q(1)*F*Y)
74 1900 T(7)=COSH(Q(2)*F*Y)
75 1900 T(8)=SINH (Q(2)*F*Y)
76 1900 W(1)=B(1,1)*T(5)
77 1900 W(2)=B(2,1)*T(6)
78 1900 W(3)=B(3,1)*T(7)
79 1900 W(4)=B(4,1)*T(8)
80 1900 W(5)=B(5,1)*T(1)*T(3)
81 1900 W(6)=B(6,1)*T(2)*T(3)
82 1900 W(7)=B(7,1)*T(1)*T(4)
83 1900 W(8)=B(8,1)*T(2)*T(4)
84 1900 W(9)=W(1)+W(2)+W(3)+W(4)+W(5)+W(6)+W(7)+W(8)+1.0
85 1900 W=5.2*W(9)/(PM*G)**3
86 1900 XM(1)=-W(9)
87 1900 XM(2)=Q(1)+2*(B(1,1)*T(5)+B(2,1)*T(6))
88 1900 XM(3)=Q(2)+2*(B(1,3)*T(7)+B(4,1)*T(8))
89 1900 WK=W(5)+W(6)+W(7)+W(8)
90 1900 XM(4)=(Q(3)+2*(Q(4)*T(2)*WK))
91 1900 XM(5)=2*(Q(3)*Q(4)*(-B(5,1)*T(2)*T(4)-B(6,1)*T(1)*T(4)+B(7,1)*
92 1900 T(3))+B(8,1)*T(1)*T(3))
93 1900 ZM=(XM(2)+XM(3)+XM(4)+XM(5))/((PM*G)**2
94 1900 XM=- (XM+3*(ZM+1.0)/(2+1*PM*G))

```

FORTRAN SOURCE LIST

SUBROUTINE STATEMENT

```

42 YMM=-{ZM+3*XN(1)+1.}/{2.1*PM*6}
43 XYM(1)=G(1)*(B(2,1)*T(6)+B(2,1)*T(5))
44 XYM(2)=G(2)*(B(3,1)*T(6)+B(4,1)*T(7))
45 XYM(3)=G(3)*(B(5,1)*T(2)*T(3)+B(6,1)*T(1)*T(5))
46 XYM(4)=G(5)*(B(7,1)*T(2)*T(4)+B(5,1)*T(1)*T(4))
47 XYM(5)=G(4)*(-B(5,1)*T(1)*T(4)-B(5,1)*T(2)*T(6))
48 XYM(6)=G(4)*(-B(5,1)*T(1)*T(4)-B(5,1)*T(2)*T(4))
49 XYM(7)=G(4)*(B(7,1)*T(1)*T(2)+B(8,1)*T(2)*T(5))
50 XYM(8)=XYM(1)+XYM(2)+XYM(3)+XYM(4)+XYM(5)+XYM(6)
51 XKM=-XYM(7)/(4.0*(PM*6)**2)
52 PRINT2000,Y,WW,XMN,XYM,XKM
53 2000 FORMAT(1X,5E15.4)
54 Y=Y+.1
55 IF(Y-1.0)1950,1900,2100
56 1900 F=F+.2
57 IF(F-2.178,78,2200
58 2200 X=X+.0
59 IF(X-50.0)2,2,6
60 PM=PM+2.0
61 IF(PM-7.0)4,4,9
62 4 STOP
63 9 END

```

IBMAP ASSEMBLY

MESSAGES FOR ABOVE ASSEMBLY

FORTRAN SOURCE LIST

SOURCE STATEMENT

```

3 .IBMAP CUBIC 1000CK
4      SUBROUTINE CUBIC(A,XR,XI)
5      DIMENSION A(4),XR(3),XI(2)
6      IPATH=2
7      EX=1.0/2.
8      1000 FORMAT(1F14.2)
9      IF(A(4))1006,1004,1006
10     1004 XR(1)=0.
11     GO TO 1034
12     1006 A=A(1)**2
13     C=(27.*A2*A(4)+9.*A(1)*A(2)*A(3)+2.*A(2)**3)/(54.*A2*A(1))
14     IF(C)1010,1002,1014
15     1002 Z=0.
16     GO TO 1032
17     1010 Q=-0
18     IPATH=1
19     1014 P=(3.*A(1)*A(3)-A(2)**2)/(9.*A2)
20     ARG=P**3+Q**2
21     IF(ARG)1016,1018,1020
22     1016 Z=-Z**SQR(-P)*COS(ATAN(SQR(-ARG)/Q)/3.)
23     GO TO 1028
24     1018 Z=-Z**Q**EX
25     GO TO 1028
26     1020 SARG=SQR(ARG)
27     IF(P)1022,1024,1026
28     1022 Z=-(Q+SARG)**EX-(Q-SARG)**EX
29     GO TO 1026
30     1024 Z=-(2.*Q)**EX
31     GO TO 1028
32     1026 Z=(SARG-Q)**EX-(SARG+Q)**EX
33     1028 GO TO 1030,1032,IPATH
34     1030 Z=-Z
35     1032 XR(1)=(3.*A(1)*Z-A(2))/(3.*A(1))
36     1034 AQ(1)=A(1)
37     AQ(2)=A(2)+XR(1)*A(1)
38     AQ(3)=A(3)+XR(1)*AQ(2)
39     CALL QUAD(AQ,KR(2),XR(3),XI)
40     RETURN
41     END

```

IBMAP ASSEMBLY CUBIC

MESSAGES FOR ABOVE ASSEMBLY

FORTRAN SOURCE LIST

500 SOURCE STATEMENT

```

C 24 IBMAP QUAD  NODECK
1 25 EXECUTING SUBR(A,X1,X2,X1)
2 26 DIMENSION A(3)
3 27 FORMAT(5F14.3)
4 28 X1=-A(2)/(P.+A(1))
5 29 D15C=X1**2-A(3)/A(1)
6 30 IF(D15C)27,28,29
7 31 X2=SQRT(-D15C)
8 32 X1=X1
9 33 X2=X2
10 34 GO TO 29
11 35 X3=X1+X2
12 36 X32=X1-X2
13 37 X1=0.
14 38 X2=0.
15 39 RETURN
21 40 END

```

IBMAP ASSEMBLY QUAD

MESSAGES FOR ABOVE ASSEMBLY

1BLDR -- JOB 000000

* IS BEING ENTERED INTO STORAGE.

APPENDIX 'D'

Tabulation of Deflection and Moments for different
Values of Parameter a^2/hR_1 and b/a ratio

$a^2/hR_1 = 0.5$ & $b/a = 1.0$		$a/a = 0.5$	$R_1/R_2 = 0.5$	
y/b	$10^3 w_0/E^1 T_0^2$	$10^3 M_{xy}/E^1 T_0^2$	$10^3 M_y/E^1 T_0^2$	$10^3 M_{xy}/E^1 T_0^2$
0.0	00.00	-10.61	0.0	-47.61
0.1	27.33	-18.97	-20.36	-56.36
0.2	51.16	-22.42	-25.34	-39.04
0.3	65.91	-23.25	-24.07	-21.01
0.4	72.95	-23.11	-21.65	-8.33
0.5	74.92	-22.95	-20.60	0.00
0.6	72.96	-23.11	-21.65	8.33
0.7	65.91	-23.25	-24.07	21.01
0.8	51.16	-22.42	-25.34	39.04
0.9	27.33	-18.97	-20.36	56.36
1.0	0.00	-10.61	0.00	47.61
 $a^2/hR_1 = 0.5$ & $b/a = 1.2$				
y/b	$10^3 w_0/E^1 T_0^2$	$10^3 M_{xy}/E^1 T_0^2$	$10^3 M_y/E^1 T_0^2$	$10^3 M_{xy}/E^1 T_0^2$
0.0	00.00	-10.61	000.00	-38.48
0.1	27.23	-19.89	-22.94	-43.63
0.2	46.94	-21.93	-24.85	-21.71
0.3	56.16	-21.37	-21.08	-04.33
0.4	54.44	-20.20	-17.04	1.62
0.5	63.75	-19.68	-15.40	0.00
0.6	54.44	-20.20	-17.40	-1.62
0.7	54.16	-21.37	-21.02	8.33
0.8	46.94	-21.93	-24.85	21.71
0.9	27.23	-19.89	-22.94	43.63
1.0	00.00	-10.61	00.00	38.48

$a^2/bR_1=0$ $b/a=104$		$x/a=0.5$	$R_1/R_2=0.5$	
y/b	$10^3 \mu_B/T\text{e}^2$	$10^2 M_x/E^2 \text{Th}^2$	$10^2 M_y/E^2 \text{Th}^2$	$10^2 M_{xy}/E^2 \text{Th}^2$
0.0	0.0	-10.61	0.00	-34.73
0.1	28.63	-20.15	-23.96	-36.06
0.2	46.02	-21.55	-24.41	-3.35
0.3	43.06	-19.83	-18.91	6.96
0.4	36.34	-17.89	-14.32	6.80
0.5	35.00	-17.13	-12.67	-6.81
0.6	36.34	-17.93	-14.32	-6.26
0.7	43.07	-19.83	-18.91	6.35
0.8	44.03	-21.55	-24.41	36.07
0.9	28.64	-20.15	-23.96	34.75
1.0	0.00	-10.61	0.00	34.75
$a^2/bR_1=5.0$	$b/a=1.6$	$x/a=0.5$	$R_1/R_2=0.5$	
y/b	$10^3 \mu_B/T\text{e}^2$	$10^2 M_x/E^2 \text{Th}^2$	$10^2 M_y/E^2 \text{Th}^2$	$10^2 M_{xy}/E^2 \text{Th}^2$
0.0	0.0	-10.61	0.00	-33.33
0.1	30.76	-20.70	-28.19	-30.34
0.2	41.54	-21.10	-23.57	-1.88
0.3	32.76	-18.47	-17.23	16.59
0.4	20.19	-16.13	-12.89	12.86
0.5	14.72	-15.27	-11.48	0.00
0.6	20.19	-15.13	-12.69	-12.86
0.7	32.76	-18.47	-17.23	-16.59
0.8	41.54	-21.10	-23.57	-1.88
0.9	30.76	-20.70	-28.19	30.36
1.0	0.00	-10.61	0.00	33.33
$a^2/bR_1=5.0$	$b/a=1.8$	$x/a=0.5$	$R_1/R_2=0.5$	
y/b	$10^3 \mu_B/T\text{e}^2$	$10^2 M_x/E^2 \text{Th}^2$	$10^2 M_y/E^2 \text{Th}^2$	$10^2 M_{xy}/E^2 \text{Th}^2$
0.0	0.00	-10.61	0.00	-32.66
0.1	32.73	-21.07	-26.90	-26.06
0.2	36.37	-20.46	-22.32	9.72
0.3	22.78	-17.92	-15.80	21.10
0.4	6.13	-14.80	-12.28	14.14
0.5	-0.69	-15.97	-11.33	0.00
0.6	6.13	-14.80	-12.28	-14.14
0.7	22.78	-17.23	-15.80	-21.11
0.8	36.39	-20.66	-22.32	-9.72
0.9	32.74	-21.09	-26.91	26.06
1.0	0.00	-10.62	0.00	32.65

$a^2/hR_1=5.0$ $b/a=2.0$		$a/a=0.5$ $R_1/R_2=0.5$		
Y/b	$10^3 R_0/2T_0^2$	$10^3 M_0/E^2 T_0^2$	$10^3 M_0/E^2 T_0^2$	$10^3 M_0/E^2 T_0^2$
0.0	0.00	-10.61	0.00	-31.56
0.1	38.08	-21.27	-26.18	-19.71
0.2	33.97	-19.65	-20.82	15.71
0.3	12.84	-16.05	-14.60	22.69
0.4	-5.76	-13.78	-12.18	13.36
0.5	-12.51	-13.09	-11.70	0.00
0.6	-5.77	-13.78	-12.18	-13.36
0.7	12.82	-16.04	-14.60	-22.69
0.8	33.95	-19.65	-20.82	-15.71
0.9	34.08	-21.27	-26.18	15.36
1.0	0.00	-10.61	0.00	31.56
$a^2/hR_1=10.0$ $b/a=1.0$		$a/a=0.5$ $R_1/R_2=0.5$		
Y/b	$10^3 R_0/2T_0^2$	$10^3 M_0/E^2 T_0^2$	$10^3 M_0/E^2 T_0^2$	$10^3 M_0/E^2 T_0^2$
0.0	0.00	-10.61	0.00	-20.96
0.1	14.37	-17.35	-16.84	-30.36
0.2	26.80	-19.48	-22.13	-16.85
0.3	32.89	-19.34	-20.08	-6.81
0.4	34.62	-18.67	-17.36	-1.16
0.5	34.80	-18.35	-16.24	0.00
0.6	34.62	-18.67	-17.36	1.16
0.7	32.89	-19.34	-20.08	6.81
0.8	26.80	-19.48	-22.13	18.56
0.9	14.37	-17.35	-16.84	30.36
1.0	0.00	-10.61	0.00	20.96
$a^2/hR_1=10.0$ $b/a=1.2$		$a/a=0.5$ $R_1/R_2=0.5$		
Y/b	$10^3 R_0/2T_0^2$	$10^3 M_0/E^2 T_0^2$	$10^3 M_0/E^2 T_0^2$	$10^3 M_0/E^2 T_0^2$
0.0	0.00	-10.61	0.00	-24.05
0.1	13.10	-17.78	-20.30	-21.34
0.2	21.04	-18.87	-21.77	-4.86
0.3	20.12	-17.59	-17.76	6.13
0.4	15.91	-16.13	-14.04	6.60
0.5	13.63	-15.55	-12.66	0.00
0.6	15.91	-16.13	-14.04	-6.40
0.7	20.12	-17.59	-17.76	-6.13
0.8	21.04	-18.87	-21.77	-4.86
0.9	13.10	-17.78	-20.30	21.34
1.0	0.00	-10.61	0.00	24.05

$a^2/bR_1=10$	$b/a=1.4$	$x/a=0.8$	$R_y/R_g=0.5$	
Y/b	$10^3mb/\perp T\text{e}^2$	$10^3Mx/\perp T\text{e}^2$	$10^3My/\perp T\text{e}^2$	$10^3Mxy/\perp T\text{e}^2$
0.0	0.00	-10.61	0.00	-10.72
0.1	12.66	-18.18	-21.66	-15.10
0.2	15.85	-18.28	-21.13	5.47
0.3	7.83	-16.06	-16.02	15.02
0.4	-1.85	-14.14	-12.27	11.08
0.5	-5.91	-13.43	-11.02	0.00
0.6	-1.86	-14.14	-12.72	-11.08
0.7	7.83	-16.06	-16.02	-15.02
0.8	15.85	-18.28	-21.13	-5.47
0.9	12.66	-18.18	-21.66	-15.10
1.0	0.00	-10.61	0.00	10.72
$a^2/bR_1=10$	$b/a=1.6$	$x/a=0.8$	$R_y/R_g=0.5$	
Y/b	$10^3mb/\perp T\text{e}^2$	$10^3Mx/\perp T\text{e}^2$	$10^3My/\perp T\text{e}^2$	$10^3Mxy/\perp T\text{e}^2$
0.0	0.00	-10.61	0.00	-8.75
0.1	19.37	-18.38	-27.60	-9.94
0.2	10.46	-17.89	-20.07	13.17
0.3	-3.94	-14.67	-14.61	20.99
0.4	-17.72	-12.60	-31.49	12.98
0.5	-23.08	-11.90	-10.60	0.00
0.6	-17.72	-12.60	-11.49	-12.98
0.7	-3.94	-14.67	-14.61	-20.99
0.8	10.46	-17.89	-20.07	-13.17
0.9	19.37	-18.38	-27.60	-9.94
1.0	0.00	-10.61	0.00	8.75
$a^2/bR_1=10$	$b/a=1.8$	$x/a=0.8$	$R_y/R_g=0.5$	
Y/b	$10^3mb/\perp T\text{e}^2$	$10^3Mx/\perp T\text{e}^2$	$10^3My/\perp T\text{e}^2$	$10^3Mxy/\perp T\text{e}^2$
0.0	0.00	-10.61	0.00	-7.17
0.1	11.76	-18.44	-25.89	-6.07
0.2	4.33	-16.59	-18.74	18.88
0.3	-18.41	-13.38	-13.46	22.36
0.4	-31.69	-11.39	-11.26	12.82
0.5	-37.36	-10.78	-10.62	0.00
0.6	-31.68	-11.39	-11.26	-12.82
0.7	-18.41	-13.38	-13.46	-22.36
0.8	4.33	-16.59	-18.74	-18.88
0.9	11.76	-18.44	-25.89	6.07
1.0	0.00	-10.61	0.00	7.17

$a^2/bR_1=10$	$b/a=2.0$	$x/a=0.5$	$R_y/R_x=0.5$	
y/b	$10^3 M_b/2\pi T_b^2$	$10^3 M_x/2\pi T_b^2$	$10^3 M_y/2\pi T_b^2$	$10^3 M_{xy}/2\pi T_b^2$
0.0	0.00	-10.61	0.00	- 5.73
0.1	10.66	-18.37	-22.98	- 6.32
0.2	-2.62	-18.59	-17.33	22.59
0.3	-26.86	-12.19	-12.56	22.51
0.4	-43.86	-10.42	-11.31	11.46
0.5	-49.82	- 9.93	-11.26	0.00
0.6	-43.86	-10.42	-11.31	-11.46
0.7	-26.86	-12.19	-12.56	-22.51
0.8	- 2.62	-15.59	-17.33	-22.59
0.9	10.67	-18.37	-22.95	6.32
1.0	0.00	-10.61	0.00	5.73
$a^2/bR_1=15$	$b/a=2.0$	$x/a=0.5$	$R_y/R_x=0.5$	
y/b	$10^3 M_b/2\pi T_b^2$	$10^3 M_x/2\pi T_b^2$	$10^3 M_y/2\pi T_b^2$	$10^3 M_{xy}/2\pi T_b^2$
0.0	0.00	-10.61	0.00	2.00
0.1	3.11	-14.39	-16.76	- 8.01
0.2	5.92	-16.93	-19.14	- 1.39
0.3	4.01	-15.99	-16.63	4.61
0.4	2.31	-14.97	-13.39	4.46
0.5	1.12	-16.56	-12.32	0.60
0.6	2.31	-14.97	-13.89	- 4.45
0.7	4.91	-16.99	-16.63	- 4.61
0.8	6.86	-16.03	-19.14	1.59
0.9	3.11	-14.67	-16.76	8.61
1.0	0.00	-10.61	0.00	- 2.00
$a^2/bR_1=15$	$b/a=1.5$	$x/a=0.5$	$R_y/R_x=0.5$	
y/b	$10^3 M_b/2\pi T_b^2$	$10^3 M_x/2\pi T_b^2$	$10^3 M_y/2\pi T_b^2$	$10^3 M_{xy}/2\pi T_b^2$
0.0	0.00	-10.61	0.00	7.60
0.1	0.56	-16.13	-18.26	- 1.53
0.2	- 1.49	-16.11	-18.74	6.92
0.3	- 6.75	-18.39	-14.89	11.90
0.4	-16.08	-12.02	-11.79	9.50
0.5	-19.04	-12.36	-10.71	0.00
0.6	-16.08	-12.02	-11.79	- 6.50
0.7	- 6.75	-14.36	-14.89	-11.90
0.8	- 1.49	-16.11	-18.74	- 6.92
0.9	0.56	-16.13	-18.26	7.60
1.0	0.00	-10.61	0.00	- 7.60

$a^2/bR_1=15$ $b/a=1.4$		$x/a=0.5$	$R_1/R_2=0.5$	
y/b	$10^3 \text{Wh}/\text{ATe}^2$	$10^3 \text{Mx}/\text{ATe}^2$	$10^3 \text{My}/\text{ATe}^2$	$10^3 \text{Mxy}/\text{ATe}^2$
0.0	0.00	-10.61		
0.1	-1.66	-16.14	0.00	11.05
0.2	-6.74	-15.26	-19.24	3.29
0.3	-21.83	-15.88	-17.91	16.36
0.4	-33.04	-11.15	-15.56	19.18
0.5	-37.28	-10.67	-10.86	11.86
0.6	-33.04	-11.15	-10.06	0.00
0.7	-21.83	-15.88	-10.06	-11.06
0.8	-6.74	-15.26	-13.64	-19.18
0.9	-1.66	-16.14	-17.91	-16.36
1.0	0.00	-10.61	-19.24	-3.29
			0.00	-11.05

$a^2/bR_1=15$ $b/a=1.6$		$x/a=0.5$	$R_1/R_2=0.5$	
y/b	$10^3 \text{Wh}/\text{ATe}^2$	$10^3 \text{Mx}/\text{ATe}^2$	$10^3 \text{My}/\text{ATe}^2$	$10^3 \text{Mxy}/\text{ATe}^2$
0.0	0.00	-10.61	0.00	15.38
0.1	-3.98	-16.21	-19.76	7.72
0.2	-16.32	-14.30	-16.78	21.61
0.3	-34.34	-11.53	-12.47	21.48
0.4	-48.01	-9.84	-10.60	19.09
0.5	-52.89	-9.32	-10.21	0.00
0.6	-48.01	-9.84	-10.21	-12.09
0.7	-34.34	-11.53	-12.47	-21.48
0.8	-16.32	-14.30	-16.78	-21.60
0.9	-3.98	-16.21	-19.76	-7.71
1.0	0.00	-10.61	0.00	-15.38

$a^2/bR_1=15$ $b/a=1.8$		$x/a=0.5$	$R_1/R_2=0.5$	
y/b	$10^3 \text{Wh}/\text{ATe}^2$	$10^3 \text{Mx}/\text{ATe}^2$	$10^3 \text{My}/\text{ATe}^2$	$10^3 \text{Mxy}/\text{ATe}^2$
0.0	0.00	-10.61	0.00	15.26
0.1	-6.54	-16.04	-19.90	11.76
0.2	-24.30	-15.27	-15.64	26.17
0.3	-46.22	-10.31	-11.67	21.71
0.4	-61.05	-8.78	-10.65	11.10
0.5	-68.98	-8.36	-10.60	0.00
0.6	-61.05	-8.78	-10.65	-11.10
0.7	-46.22	-10.31	-11.67	-21.70
0.8	-24.30	-15.27	-15.64	-26.16
0.9	-6.54	-16.04	-19.90	-11.79
1.0	0.00	-10.61	0.00	-15.21

$a^2/bR_1=16$ $b/a=2.0$		$x/a=0.5$	$R_1/R_2=0.5$	
y/b	$10^3 M_b/\text{e}^2 \text{Ta}^2$	$10^3 M_x/\text{e}^2 \text{Th}^2$	$10^3 M_y/\text{e}^2 \text{Th}^2$	$10^3 M_{xy}/\text{e}^2 \text{Th}^2$
0.0	0.00	-10.61	0.00	16.79
0.1	-9.36	-15.70	-19.80	16.53
0.2	-32.56	-12.24	-14.37	17.40
0.3	-57.37	-9.23	-11.14	20.73
0.4	-72.41	-7.89	-10.61	9.65
0.5	-77.10	-7.56	-10.98	0.00
0.6	-72.41	-7.89	-10.81	-9.65
0.7	-57.37	-9.23	-11.14	-20.73
0.8	-32.56	-12.24	-14.37	-27.39
0.9	-9.36	-15.70	-19.81	-16.51
1.0	0.00	-10.61	0.00	-16.69
$a^2/bR_1=20$ $b/a=1.0$		$x/a=0.5$	$R_1/R_2=0.5$	
y/b	$10^3 M_b/\text{e}^2 \text{Ta}^2$	$10^3 M_x/\text{e}^2 \text{Th}^2$	$10^3 M_y/\text{e}^2 \text{Th}^2$	$10^3 M_{xy}/\text{e}^2 \text{Th}^2$
0.0	0.00	-10.61	0.00	16.47
0.1	-4.69	-16.82	-16.33	5.98
0.2	-8.49	-16.97	-16.88	9.87
0.3	-14.07	-13.73	-16.24	11.66
0.4	-19.09	-12.86	-11.73	7.67
0.5	-21.08	-12.12	-10.80	0.00
0.6	-19.09	-12.56	-11.73	-7.67
0.7	-14.07	-13.73	-14.24	-11.66
0.8	-8.49	-16.97	-16.88	-9.87
0.9	-4.69	-16.82	-16.33	-5.98
1.0	0.00	-10.61	0.00	-16.47
$a^2/bR_1=20$ $b/a=1.2$		$x/a=0.6$	$R_1/R_2=0.5$	
y/b	$10^3 M_b/\text{e}^2 \text{Ta}^2$	$10^3 M_x/\text{e}^2 \text{Th}^2$	$10^3 M_y/\text{e}^2 \text{Th}^2$	$10^3 M_{xy}/\text{e}^2 \text{Th}^2$
0.0	0.00	-10.61	0.00	23.45
0.1	-8.34	-14.89	-16.85	11.72
0.2	-17.13	-16.11	-16.36	17.73
0.3	-28.19	-12.16	-12.90	17.96
0.4	-37.05	-10.71	-10.80	10.83
0.5	-40.36	-10.21	-9.75	0.00
0.6	-37.05	-10.71	-10.80	-10.83
0.7	-28.19	-12.16	-12.90	-17.96
0.8	-17.13	-16.11	-16.36	-17.73
0.9	-8.34	-14.89	-16.85	-12.72
1.0	0.00	-10.61	0.00	-23.45

$a^2/bR_1=20$ $b/a=1.4$		$x/a=0.5$	$R_1/R_2=0.5$	
y/b	$10^3 M_h/1T_{\mu}^2$	$10^2 M_x/1T_{\mu}^2$	$10^2 M_y/1T_{\mu}^2$	$10^3 M_{xy}/1T_{\mu}^2$
0.0	0.00	-10.61	0.00	26.77
0.1	-11.91	-14.81	-17.25	16.13
0.2	-26.83	-13.12	-15.45	23.07
0.3	-41.69	-10.74	-11.86	20.91
0.4	-53.26	-9.27	-10.13	11.65
0.5	-57.36	-8.81	-9.72	0.00
0.6	-53.26	-9.27	-10.13	-11.65
0.7	-41.69	-10.74	-11.86	-20.91
0.8	-26.83	-13.12	-15.45	-23.07
0.9	-11.91	-14.81	-17.25	-26.77
1.0	0.00	-10.61	0.00	-26.77
$a^2/bR_1=20$ $b/a=1.6$		$x/a=0.5$	$R_1/R_2=0.5$	
y/b	$10^3 M_h/1T_{\mu}^2$	$10^2 M_x/1T_{\mu}^2$	$10^2 M_y/1T_{\mu}^2$	$10^3 M_{xy}/1T_{\mu}^2$
0.0	0.00	-10.61	0.00	29.23
0.1	-15.58	-14.60	-17.55	19.86
0.2	-34.66	-12.07	-14.37	26.82
0.3	-54.36	-9.47	-11.11	21.82
0.4	-67.44	-8.11	-10.17	11.04
0.5	-71.81	-7.73	-10.10	0.00
0.6	-67.44	-8.11	-10.17	-11.04
0.7	-54.36	-9.47	-11.11	-21.82
0.8	-34.66	-12.07	-14.37	-26.82
0.9	-15.58	-14.60	-17.55	-29.23
1.0	0.00	-10.61	0.00	-29.23
$a^2/bR_1=20$ $b/a=1.8$		$x/a=0.5$	$R_1/R_2=0.5$	
y/b	$10^3 M_h/1T_{\mu}^2$	$10^2 M_x/1T_{\mu}^2$	$10^2 M_y/1T_{\mu}^2$	$10^3 M_{xy}/1T_{\mu}^2$
0.0	0.00	-10.61	0.00	31.11
0.1	-19.39	-14.29	-17.86	23.08
0.2	-43.49	-11.03	-13.33	26.50
0.3	-66.10	-8.36	-10.64	20.76
0.4	-79.74	-7.15	-10.35	9.82
0.5	-84.05	-6.84	-10.49	0.00
0.6	-79.74	-7.15	-10.35	-9.82
0.7	-66.09	-8.36	-10.64	-20.77
0.8	-43.49	-11.03	-13.34	-26.50
0.9	-19.39	-14.29	-17.86	-23.08
1.0	0.00	-10.60	0.00	-31.11

$a^2/bR_1=20$	$b/a=2.0$	$x/a=0.5$	$R_1/R_2=0.5$	
y/b	$10^3 M_y/T\text{e}^2$	$10^2 M_y/E/T\text{h}^2$	$10^2 M_y/E/T\text{h}^2$	$10^3 M_{xy}/E/T\text{h}^2$
0.0	0.00	-10.61	0.00	32.56
0.1	-23.11	-13.92	-17.36	28.87
0.2	-52.18	-10.02	-12.62	29.34
0.3	-76.77	-7.14	-10.41	19.30
0.4	-90.39	-6.34	-10.58	8.83
0.5	-94.51	-6.06	-10.77	0.00
0.6	-90.39	-6.34	-10.58	-8.83
0.7	-76.75	-7.41	-10.40	-19.32
0.8	-52.09	-10.04	-12.41	-29.40
0.9	-22.96	-13.94	-17.40	-56.92
1.0	0.00	-10.68	0.00	-31.89

Two Edges simply supported and two edges clamped

$a^2/bR_1=5$	$b/a=1.0$	$x/a=0.5$	$R_1/R_2=0.5$	
y/b	$10^3 M_y/T\text{e}^2$	$10^2 M_y/E/T\text{h}^2$	$10^2 M_y/E/T\text{h}^2$	$10^3 M_{xy}/E/T\text{h}^2$
0.0	0.00	-4.27	21.12	0.00
0.1	-13.02	-16.67	-9.61	-61.43
0.2	34.24	-19.71	-20.96	-40.14
0.3	50.92	-21.80	-23.33	-36.37
0.4	60.42	-22.40	-22.74	-12.53
0.5	63.40	-22.49	-22.23	0.00
0.6	60.42	-22.46	-22.74	12.23
0.7	50.92	-21.80	-23.33	36.37
0.8	34.24	-19.71	-20.96	40.14
0.9	13.02	-16.67	-9.61	-61.43
1.0	0.00	-4.27	21.12	0.00

$a^2/bR_1=5$	$b/a=1.2$	$x/a=0.5$	$R_1/R_2=0.5$	
y/b	$10^3 M_y/T\text{e}^2$	$10^2 M_y/E/T\text{h}^2$	$10^2 M_y/E/T\text{h}^2$	$10^3 M_{xy}/E/T\text{h}^2$
0.0	0.00	-5.50	17.03	0.00
0.1	14.74	-16.31	-14.64	-35.91
0.2	33.98	-20.12	-22.37	-26.62
0.3	44.44	-20.68	-21.03	-10.26
0.4	47.72	-19.96	-18.09	-2.08
0.5	48.17	-19.62	-16.81	0.00
0.6	47.72	-19.96	-18.09	2.08
0.7	44.44	-20.58	-21.03	10.26
0.8	33.98	-20.12	-22.37	26.62
0.9	14.74	-16.31	-14.64	-35.91
1.0	0.00	-5.50	17.03	0.00

$a^2/bR_1=5$	$b/a=1.4$	$x/a=0.5$	$R_y/R_g=0.5$	
Y/b	$10^3 \text{m}^2/\text{e} \cdot \text{T} \cdot \text{s}^2$	$10^3 \text{Mx}/\text{e} \cdot \text{T} \cdot \text{s}^2$	$10^3 \text{My}/\text{e} \cdot \text{T} \cdot \text{s}^2$	$10^3 \text{Mxy}/\text{e} \cdot \text{T} \cdot \text{s}^2$
0.0	0.00	- 6.04	15.21	0.00
0.1	16.70	- 17.41	- 18.08	- 31.54
0.2	35.98	- 20.21	- 22.96	- 13.19
0.3	55.87	- 19.32	- 19.21	2.87
0.4	32.13	- 17.79	- 15.14	5.80
0.5	29.88	- 17.14	- 13.88	0.00
0.6	32.14	- 17.80	- 15.14	- 8.80
0.7	35.87	- 19.32	- 19.21	- 2.88
0.8	32.99	- 20.21	- 22.96	13.19
0.9	16.70	- 17.41	- 18.08	33.04
1.0	0.00	- 6.04	15.21	0.00

$a^2/bR_1=5$	$b/a=1.6$	$x/a=0.5$	$R_y/R_g=0.5$	
Y/b	$10^3 \text{m}^2/\text{e} \cdot \text{T} \cdot \text{s}^2$	$10^3 \text{Mx}/\text{e} \cdot \text{T} \cdot \text{s}^2$	$10^3 \text{My}/\text{e} \cdot \text{T} \cdot \text{s}^2$	$10^3 \text{Mxy}/\text{e} \cdot \text{T} \cdot \text{s}^2$
0.0	0.00	- 6.25	14.51	0.00
0.1	18.90	- 18.26	- 20.40	- 27.97
0.2	31.70	- 20.05	- 22.80	- 3.26
0.3	27.01	- 18.13	- 17.67	12.06
0.4	17.04	- 16.05	- 13.46	10.70
0.5	19.49	- 18.24	- 12.02	0.00
0.6	17.30	- 16.05	- 13.46	- 10.70
0.7	27.00	- 18.13	- 17.67	- 12.06
0.8	31.69	- 20.05	- 22.79	3.27
0.9	18.89	- 18.25	- 13.39	27.96
1.0	0.00	- 6.25	14.51	0.00

$a^2/bR_1=5$	$b/a=1.8$	$x/a=0.5$	$R_y/R_g=0.5$	
Y/b	$10^3 \text{m}^2/\text{e} \cdot \text{T} \cdot \text{s}^2$	$10^3 \text{Mx}/\text{e} \cdot \text{T} \cdot \text{s}^2$	$10^3 \text{My}/\text{e} \cdot \text{T} \cdot \text{s}^2$	$10^3 \text{Mxy}/\text{e} \cdot \text{T} \cdot \text{s}^2$
0.0	0.00	- 6.37	14.11	0.00
0.1	20.99	- 18.93	- 21.97	- 24.42
0.2	29.66	- 19.66	- 22.03	4.61
0.3	18.0	- 16.98	- 16.26	17.63
0.4	3.83	- 16.69	- 12.64	12.60
0.5	- 2.64	- 13.88	- 11.57	0.00
0.6	3.82	- 14.69	- 12.64	- 12.78
0.7	18.08	- 16.98	- 16.29	- 17.59
0.8	29.59	- 19.65	- 22.03	- 4.86
0.9	50.92	- 18.91	- 21.92	24.41
1.0	0.00	- 6.37	14.11	0.00

$a^2/bR_1=5$	$b/a=2.0$	$x/a=0.5$	$R_1/R_2=0.5$	
x/b	$10^3 M_b/\mu\text{Tesla}^2$	$10^2 M_x/\text{e}^2\text{Th}^2$	$10^2 M_y/\text{e}^2\text{Th}^2$	$10^3 M_{xy}/\text{e}^2\text{Th}^2$
0.0	0.00	- 6.49	13.72	0.00
0.1	22.68	- 19.38	- 23.00	- 20.48
0.2	26.43	- 19.05	- 20.88	10.93
0.3	8.98	- 15.87	- 15.06	20.24
0.4	- 6.20	- 13.64	- 12.33	12.74
0.5	- 14.68	- 12.92	- 11.74	0.00
0.6	- 8.23	- 13.64	- 12.36	- 12.70
0.7	8.87	- 15.86	- 15.08	- 20.11
0.8	26.17	- 19.03	- 20.88	- 10.71
0.9	22.30	- 19.31	- 22.87	20.57
1.0	0.00	- 6.38	14.26	0.00
$a^2/bR_1=10$	$b/a=2.0$	$x/a=0.5$	$R_1/R_2=0.5$	
x/b	$10^3 M_b/\mu\text{Tesla}^2$	$10^2 M_x/\text{e}^2\text{Th}^2$	$10^2 M_y/\text{e}^2\text{Th}^2$	$10^3 M_{xy}/\text{e}^2\text{Th}^2$
0.0	0.00	- 7.56	10.16	0.00
0.1	8.40	- 15.40	- 13.64	- 25.31
0.2	20.24	- 19.39	- 20.37	- 20.08
0.3	27.61	- 18.68	- 19.99	- 09.82
0.4	30.64	- 18.54	- 18.03	- 3.14
0.5	31.33	- 18.32	- 17.12	0.00
0.6	30.64	- 18.54	- 18.08	3.14
0.7	27.61	- 18.88	- 19.99	9.82
0.8	20.24	- 19.39	- 20.37	20.08
0.9	8.40	- 15.40	- 13.64	25.31
1.0	0.00	- 7.56	10.16	0.00
$a^2/bR_1=10$	$b/a=1.2$	$x/a=0.5$	$R_1/R_2=0.5$	
x/b	$10^3 M_b/\mu\text{Tesla}^2$	$10^2 M_x/\text{e}^2\text{Th}^2$	$10^2 M_y/\text{e}^2\text{Th}^2$	$10^3 M_{xy}/\text{e}^2\text{Th}^2$
0.0	0.00	- 6.58	6.75	0.00
0.1	8.80	- 16.59	- 17.51	- 19.19
0.2	16.93	- 18.31	- 21.02	- 6.71
0.3	17.35	- 17.40	- 17.87	3.87
0.4	16.23	- 16.12	- 14.46	5.09
0.5	12.85	- 16.58	- 13.14	0.00
0.6	14.23	- 16.12	- 14.46	- 5.09
0.7	17.35	- 17.40	- 17.87	- 3.87
0.8	16.93	- 18.31	- 21.02	6.71
0.9	8.80	- 16.59	- 17.51	- 19.19
1.0	0.00	- 6.58	6.75	0.00

$a^2/bR_1=10$		$b/a=1.4$		$x/a=0.5$		$R_y/R_g=0.6$	
Y/b	$10^3 M_b/2\pi T a^2$	$10^2 M_x/2\pi T b^2$	$10^2 M_y/2\pi T b^2$	$10^3 M_{xy}/2\pi T b^2$	$10^3 M_{yy}/2\pi T b^2$	$10^3 M_{xy}/2\pi T b^2$	
0.0	0.00	- 9.07	5.11	0.00			
0.1	9.23	- 17.33	- 19.68	- 14.82			
0.2	12.96	- 17.92	- 20.82	3.76			
0.3	6.15	- 16.97	- 16.19	13.47			
0.4	- 8.72	- 14.14	- 12.81	10.31			
0.5	- 6.81	- 13.45	- 11.26	0.00			
0.6	- 2.72	- 14.14	- 12.81	- 10.31			
0.7	6.15	- 16.97	- 16.19	- 13.47			
0.8	12.96	- 17.92	- 20.82	- 3.76			
0.9	9.23	- 17.33	- 19.68	14.82			
1.0	0.00	- 9.07	5.11	0.00			
$a^2/bR_1=10$		$b/a=1.6$		$x/a=0.5$		$R_y/R_g=0.6$	
Y/b	$10^3 M_b/2\pi T a^2$	$10^2 M_x/2\pi T b^2$	$10^2 M_y/2\pi T b^2$	$10^3 M_{xy}/2\pi T b^2$	$10^3 M_{yy}/2\pi T b^2$	$10^3 M_{xy}/2\pi T b^2$	
0.0	0.00	- 9.36	4.16	0.00			
0.1	9.62	- 17.76	- 21.30	- 9.74			
0.2	8.35	- 17.29	- 19.99	11.74			
0.3	- 5.04	- 14.63	- 14.77	19.09			
0.4	- 18.27	- 12.59	- 11.61	12.58			
0.5	- 23.45	- 11.89	- 10.69	0.00			
0.6	- 18.27	- 12.59	- 11.61	- 12.58			
0.7	- 5.03	- 14.63	- 14.77	- 19.09			
0.8	8.35	- 17.29	- 19.99	- 11.74			
0.9	9.62	- 17.76	- 21.30	9.74			
1.0	0.00	- 9.36	4.16	0.00			
$a^2/bR_1=10$		$b/a=1.8$		$x/a=0.5$		$R_y/R_g=0.8$	
Y/b	$10^3 M_b/2\pi T a^2$	$10^2 M_x/2\pi T b^2$	$10^2 M_y/2\pi T b^2$	$10^3 M_{xy}/2\pi T b^2$	$10^3 M_{yy}/2\pi T b^2$	$10^3 M_{xy}/2\pi T b^2$	
0.0	0.00	- 9.58	3.41	0.00			
0.1	9.62	- 18.01	- 22.09	- 6.27			
0.2	2.62	- 16.47	- 18.77	17.72			
0.3	- 16.13	- 13.56	- 13.68	21.57			
0.4	- 31.96	- 11.38	- 11.32	12.66			
0.5	- 37.70	- 10.76	- 10.84	0.00			
0.6	- 31.96	- 11.38	- 11.32	- 12.65			
0.7	- 16.13	- 13.56	- 13.68	- 21.57			
0.8	2.62	- 16.47	- 18.77	- 17.71			
0.9	9.62	- 18.01	- 22.09	6.27			
1.0	0.00	- 9.58	3.41	0.00			

$a^2/bR_1=10$		$b/a=2.0$		$x/a=0.5$		$R_1/R_2=0.5$	
y/b	$10^3 \text{Wh}/\text{ATa}^2$	$10^2 \text{Mx}/\text{EATa}^2$	$10^3 \text{Wh}/\text{ATa}^2$	$10^2 \text{Mx}/\text{EATa}^2$	$10^3 \text{Wh}/\text{ATa}^2$	$10^2 \text{Mx}/\text{EATa}^2$	$10^3 \text{Wh}/\text{ATa}^2$
0.0	0.00	- 9.79	2.73	0.00	- 0.00	- 0.00	- 0.00
0.1	8.82	- 18.07	- 22.44	- 0.73	- 0.73	- 0.73	- 0.73
0.2	- 5.65	- 18.83	- 17.40	22.12	22.12	22.12	22.12
0.3	- 27.02	- 10.18	- 12.65	21.90	21.90	21.90	21.90
0.4	- 45.88	- 10.40	- 11.33	11.45	11.45	11.45	11.45
0.5	- 49.53	- 9.91	- 11.26	0.00	0.00	0.00	0.00
0.6	- 43.88	- 10.40	- 11.33	- 11.44	- 11.44	- 11.44	- 11.44
0.7	- 27.03	- 12.18	- 12.65	- 21.90	- 21.90	- 21.90	- 21.90
0.8	- 3.68	- 18.53	- 17.40	- 22.12	- 22.12	- 22.12	- 22.12
0.9	8.83	- 10.06	- 22.41	0.71	0.71	0.71	0.71
1.0	0.00	- 9.79	2.73	0.00	0.00	0.00	0.00
$a^2/bR_1=15$		$b/a=1.0$		$x/a=0.5$		$R_1/R_2=0.5$	
y/b	$10^3 \text{Wh}/\text{ATa}^2$	$10^2 \text{Mx}/\text{EATa}^2$	$10^3 \text{Wh}/\text{ATa}^2$	$10^2 \text{Mx}/\text{EATa}^2$	$10^3 \text{Wh}/\text{ATa}^2$	$10^2 \text{Mx}/\text{EATa}^2$	$10^3 \text{Wh}/\text{ATa}^2$
0.0	0.00	- 10.98	- 1.92	0.00	- 0.00	- 0.00	- 0.00
0.1	3.72	- 16.11	- 17.31	- 9.23	- 9.23	- 9.23	- 9.23
0.2	6.45	- 16.93	- 19.30	- 1.12	- 1.12	- 1.12	- 1.12
0.3	8.24	- 16.03	- 16.61	4.98	4.98	4.98	4.98
0.4	9.60	- 14.97	- 13.61	4.65	4.65	4.65	4.65
0.5	1.34	- 14.84	- 12.72	0.00	0.00	0.00	0.00
0.6	2.60	- 14.97	- 13.81	- 4.65	- 4.65	- 4.65	- 4.65
0.7	8.24	- 16.03	- 16.61	- 4.98	- 4.98	- 4.98	- 4.98
0.8	6.45	- 16.93	- 19.30	1.12	1.12	1.12	1.12
0.9	3.72	- 16.11	- 17.31	9.23	9.23	9.23	9.23
1.0	0.00	- 10.98	- 1.92	0.00	0.00	0.00	0.00
$a^2/bR_1=15$		$b/a=1.2$		$x/a=0.5$		$R_1/R_2=0.5$	
y/b	$10^3 \text{Wh}/\text{ATa}^2$	$10^2 \text{Mx}/\text{EATa}^2$	$10^3 \text{Wh}/\text{ATa}^2$	$10^2 \text{Mx}/\text{EATa}^2$	$10^3 \text{Wh}/\text{ATa}^2$	$10^2 \text{Mx}/\text{EATa}^2$	$10^3 \text{Wh}/\text{ATa}^2$
0.0	0.00	- 11.66	- 4.13	0.00	- 0.00	- 0.00	- 0.00
0.1	2.76	- 16.77	- 19.79	- 2.63	- 2.63	- 2.63	- 2.63
0.2	0.41	- 16.36	- 19.03	10.16	10.16	10.16	10.16
0.3	- 7.63	- 14.61	- 14.74	15.00	15.00	15.00	15.00
0.4	- 15.81	- 12.80	- 11.56	10.16	10.16	10.16	10.16
0.5	- 16.66	- 12.99	- 10.46	0.00	0.00	0.00	0.00
0.6	- 15.81	- 12.80	- 11.56	- 10.16	- 10.16	- 10.16	- 10.16
0.7	- 7.63	- 14.61	- 14.74	- 15.00	- 15.00	- 15.00	- 15.00
0.8	0.41	- 16.36	- 19.03	- 10.16	- 10.16	- 10.16	- 10.16
0.9	2.76	- 16.77	- 19.79	9.63	9.63	9.63	9.63
1.0	0.00	- 11.66	- 4.13	0.00	0.00	0.00	0.00

$a^2/bR_1=15$	$b/a=1.4$	$x/a=0.5$	$R_1/R_2=0.5$	
y/b	$10^3 M_b/1 T_{\text{e}}^2$	$10^3 M_x/E \text{eV}^2$	$10^3 M_y/E \text{eV}^2$	$10^3 M_{xy}/E \text{eV}^2$
0.0	0.00	~12.36	~5.84	0.00
0.1	1.54	~17.03	~20.99	5.98
0.2	~6.36	~15.50	~18.04	16.23
0.3	~20.62	~10.90	~13.30	20.83
0.4	~32.48	~11.14	~10.66	15.42
0.5	~36.90	~10.55	~9.92	0.00
0.6	~32.48	~11.14	~10.66	-12.42
0.7	~20.62	~10.90	~13.39	-20.82
0.8	~6.36	~15.03	~18.04	-16.23
0.9	1.54	~11.03	~20.99	-2.98
1.0	0.00	~12.36	~5.84	0.00
$a^2/bR_1=15$	$b/a=1.6$	$x/a=0.5$	$R_1/R_2=0.5$	
y/b	$10^3 M_b/1 T_{\text{e}}^2$	$10^3 M_x/E \text{eV}^2$	$10^3 M_y/E \text{eV}^2$	$10^3 M_{xy}/E \text{eV}^2$
0.0	0.00	~12.73	~7.07	0.00
0.1	~0.10	~17.03	~21.43	8.13
0.2	~13.86	~14.48	~16.69	23.79
0.3	~33.23	~11.53	~12.10	22.62
0.4	~47.48	~9.88	~10.47	12.41
0.5	~52.42	~9.38	~10.16	0.00
0.6	~47.48	~9.88	~10.47	-12.41
0.7	~33.23	~11.53	~12.10	-22.62
0.8	~13.86	~14.48	~16.69	-23.79
0.9	~0.09	~17.03	~21.43	-8.13
1.0	0.00	~12.73	~7.07	0.00
$a^2/bR_1=15$	$b/a=1.8$	$x/a=0.5$	$R_1/R_2=0.5$	
y/b	$10^3 M_b/1 T_{\text{e}}^2$	$10^3 M_x/E \text{eV}^2$	$10^3 M_y/E \text{eV}^2$	$10^3 M_{xy}/E \text{eV}^2$
0.0	0.00	~13.06	~8.06	0.00
0.1	~2.21	~16.88	~21.40	12.88
0.2	~21.98	~13.39	~18.29	27.42
0.3	~65.27	~10.51	~11.40	22.53
0.4	~60.44	~8.51	~10.59	11.90
0.5	~65.40	~8.42	~10.62	0.00
0.6	~60.44	~8.52	~10.59	-11.90
0.7	~65.26	~10.51	~11.39	-22.53
0.8	~21.99	~13.39	~18.29	-27.42
0.9	~2.20	~16.85	~21.40	-12.88
1.0	0.00	~13.06	~8.06	0.00

$a^2/bR_1=15$		$b/a=2.0$	$x/a=0.5$	$R_y/R_g=0.6$	
Y/b		$10^3 M_b/2\pi T_b^2$	$10^3 M_x/2\pi T_b^2$	$10^3 M_y/2\pi T_b^2$	$10^3 M_{xy}/2\pi T_b^2$
0.0		0.00	-13.28	-8.89	0.00
0.1		-4.76	-16.83	-21.04	17.41
0.2		-30.49	-12.30	-14.00	29.55
0.3		-86.86	-9.23	-10.91	21.23
0.4		-71.74	-7.95	-10.80	9.60
0.5		-76.38	-7.63	-11.03	0.00
0.6		-71.75	-7.95	-10.80	-9.60
0.7		-86.87	-9.23	-10.91	-21.23
0.8		-30.51	-12.30	-14.00	-29.55
0.9		-4.80	-16.83	-21.05	-17.41
1.0		0.00	-13.27	-8.88	0.00
$a^2/bR_1=20$		$b/a=1.0$	$x/a=0.5$	$R_y/R_g=0.6$	
Y/b		$10^3 M_b/2\pi T_b^2$	$10^3 M_x/2\pi T_b^2$	$10^3 M_y/2\pi T_b^2$	$10^3 M_{xy}/2\pi T_b^2$
0.0		0.00	-13.86	-10.83	0.00
0.1		-0.06	-16.83	-19.76	3.51
0.2		-4.27	-18.63	-17.91	12.66
0.3		-11.43	-13.84	-13.91	14.75
0.4		-17.67	-12.46	-11.02	9.39
0.5		-20.09	-11.97	-10.03	0.00
0.6		-17.67	-12.46	-11.02	-9.39
0.7		-11.43	-13.84	-13.91	-14.75
0.8		-4.27	-18.63	-17.91	-12.66
0.9		-0.06	-16.83	-19.76	-3.51
1.0		0.00	-13.86	-10.83	0.00
$a^2/bR_1=20$		$b/a=1.2$	$x/a=0.5$	$R_y/R_g=0.6$	
Y/b		$10^3 M_b/2\pi T_b^2$	$10^3 M_x/2\pi T_b^2$	$10^3 M_y/2\pi T_b^2$	$10^3 M_{xy}/2\pi T_b^2$
0.0		0.00	-16.71	-13.66	0.00
0.1		-2.24	-16.73	-21.02	10.00
0.2		-12.43	-16.63	-16.81	21.78
0.3		-28.77	-19.18	-12.30	21.17
0.4		-36.01	-10.62	-9.91	12.27
0.5		-39.75	-10.12	-9.24	0.00
0.6		-36.01	-10.62	-9.91	-12.27
0.7		-28.77	-19.18	-12.30	-21.17
0.8		-12.43	-16.63	-16.81	-21.78
0.9		-2.24	-16.73	-21.02	-10.00
1.0		0.00	-16.71	-13.66	0.00

$a^2/bR_1=20$		$b/a=1.4$	$x/a=0.5$	$R_1/R_2=0.8$	
y/b	$10^3 M_b/2T\text{e}^2$	$10^3 M_b/2\text{eTh}^2$	$10^2 M_y/2\text{eTh}^2$	$10^3 M_{xy}/2\text{eTh}^2$	
0.0	0.00	-16.28	-16.58	0.00	
0.1	-4.90	-16.57	-21.21	16.71	
0.2	-21.32	-13.45	-15.31	27.62	
0.3	-39.72	-10.70	-11.18	23.46	
0.4	-62.39	-9.23	-9.77	12.63	
0.5	-86.73	-8.81	-9.55	0.00	
0.6	-82.39	-9.23	-9.78	-12.45	
0.7	-39.72	-10.70	-11.18	-23.46	
0.8	-21.32	-13.45	-15.31	-27.62	
0.9	-4.90	-16.57	-21.21	-16.71	
1.0	0.00	-16.28	-16.58	0.00	
$a^2/bR_1=20$		$b/a=1.6$	$x/a=0.5$	$R_1/R_2=0.8$	
y/b	$10^3 M_b/2T\text{e}^2$	$10^3 M_b/2\text{eTh}^2$	$10^2 M_y/2\text{eTh}^2$	$10^3 M_{xy}/2\text{eTh}^2$	
0.0	0.00	-16.72	-17.02	0.00	
0.1	-8.05	-16.19	-20.82	22.08	
0.2	-30.66	-12.23	-13.80	31.02	
0.3	-52.85	-9.41	-10.48	23.22	
0.4	-66.87	-8.16	-10.01	11.27	
0.5	-70.99	-7.80	-10.00	0.00	
0.6	-66.87	-8.14	-10.01	-11.27	
0.7	-52.85	-9.40	-10.48	-23.22	
0.8	-30.66	-12.23	-13.81	-31.02	
0.9	-8.06	-16.20	-20.83	-22.06	
1.0	0.00	-16.72	-17.03	0.00	
$a^2/bR_1=20$		$b/a=1.8$	$x/a=0.5$	$R_1/R_2=0.8$	
y/b	$10^3 M_b/2T\text{e}^2$	$10^3 M_b/2\text{eTh}^2$	$10^2 M_y/2\text{eTh}^2$	$10^3 M_{xy}/2\text{eTh}^2$	
0.0	0.00	-16.05	-18.14	0.00	
0.1	-11.63	-15.66	-20.10	26.71	
0.2	-40.10	-11.06	-12.52	32.88	
0.3	-64.92	-8.32	-10.16	21.71	
0.4	-78.80	-7.22	-10.32	9.73	
0.5	-83.04	-6.96	-10.58	0.00	
0.6	-78.80	-7.22	-10.32	-9.73	
0.7	-64.92	-8.32	-10.15	-21.70	
0.8	-40.10	-11.06	-12.61	-32.87	
0.9	-11.64	-15.67	-20.07	-26.75	
1.0	0.00	-16.03	-18.04	0.00	

$a^2/bR_1=20$	$b/a=2.0$	$x/a=0.5$	$R_1/R_2=0.5$	
y/b	$10^3 M_y/L^2 T^2$	$10^2 M_x/E L^2 T^2$	$10^2 M_y/E L^2 T^2$	$10^3 M_{xy}/E L^2 T^2$
0.0	0.00	-16.31	-19.00	0.00
0.1	-15.54	-15.09	-19.19	30.58
0.2	-49.49	-9.99	-11.52	32.78
0.3	-75.79	-7.40	-10.06	19.71
0.4	-89.38	-6.43	-10.58	8.35
0.5	-93.37	-6.20	-10.86	0.00
0.6	-89.37	-6.43	-10.58	-6.34
0.7	-75.76	-7.40	-10.06	-19.72
0.8	-49.47	-10.00	-11.84	-32.76
0.9	-15.57	-15.08	-19.16	-30.47
1.0	0.00	-16.22	-19.06	0.00

Tabulation of Deflection and Moments for various values of a^2/bR_1 and b/a ratio constant.

All edges simply supported.

$b/a=1.0$	$y/b=0.5$	$x/a=0.5$	$R_1/R_2=0.5$
a^2/bR_1	$10^3 M_y/L^2 T^2$	$10^2 M_x/E L^2 T^2$	$10^2 M_y/E L^2 T^2$
5	53.75	-19.68	-18.48
10	34.80	-16.36	-16.24
15	1.12	-14.88	-12.82
20	-21.06	-12.12	-10.80
25	-38.06	-10.64	-9.72
30	-44.01	-9.74	-9.17
35	-49.91	-9.17	-8.91
40	-53.98	-8.81	-8.79
45	-56.00	-8.56	-8.74
50	-58.08	-8.39	-8.74

Two Edges simply supported and top edges clamped

$b/a=1.0$	$y/b=0.5$	$x/a=0.5$	$R_1/R_2=0.5$
a^2/bR_1	$10^3 M_y/L^2 T^2$	$10^2 M_x/E L^2 T^2$	$10^2 M_y/E L^2 T^2$
5	63.40	-22.49	-22.24
10	31.33	-18.32	-17.12
15	1.34	-14.84	-12.72
20	-20.09	-11.97	-10.03
25	-34.26	-10.40	-8.68
30	-43.81	-9.48	-8.16
35	-49.64	-8.98	-8.07
40	-53.79	-8.64	-8.19
45	-56.68	-8.36	-8.38
50	-58.76	-8.35	-8.55

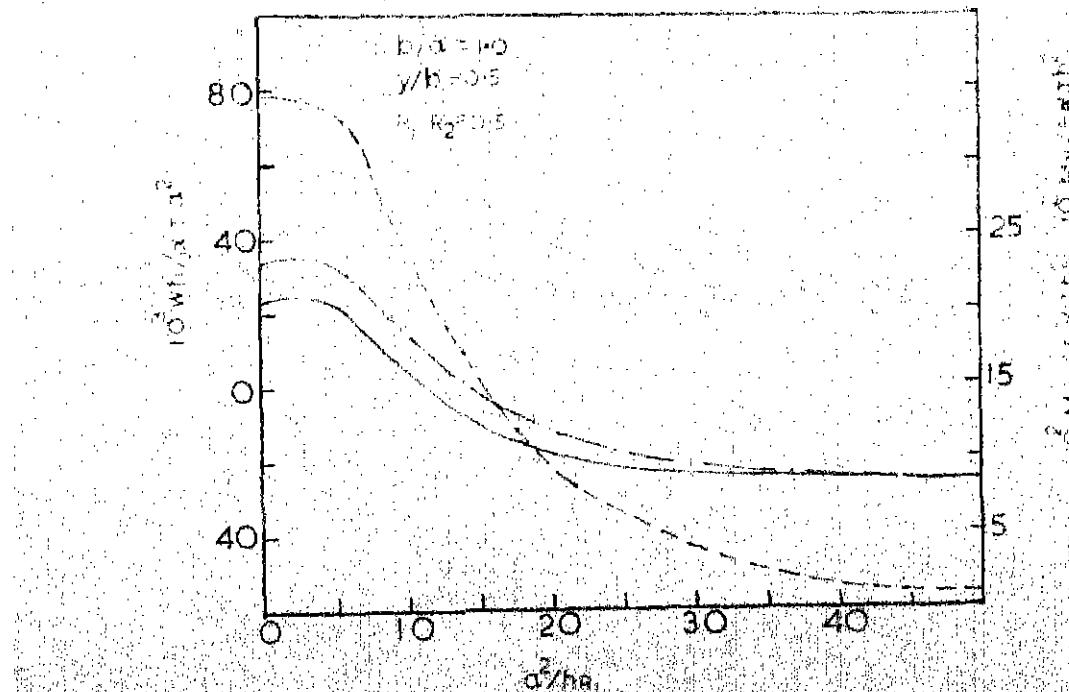
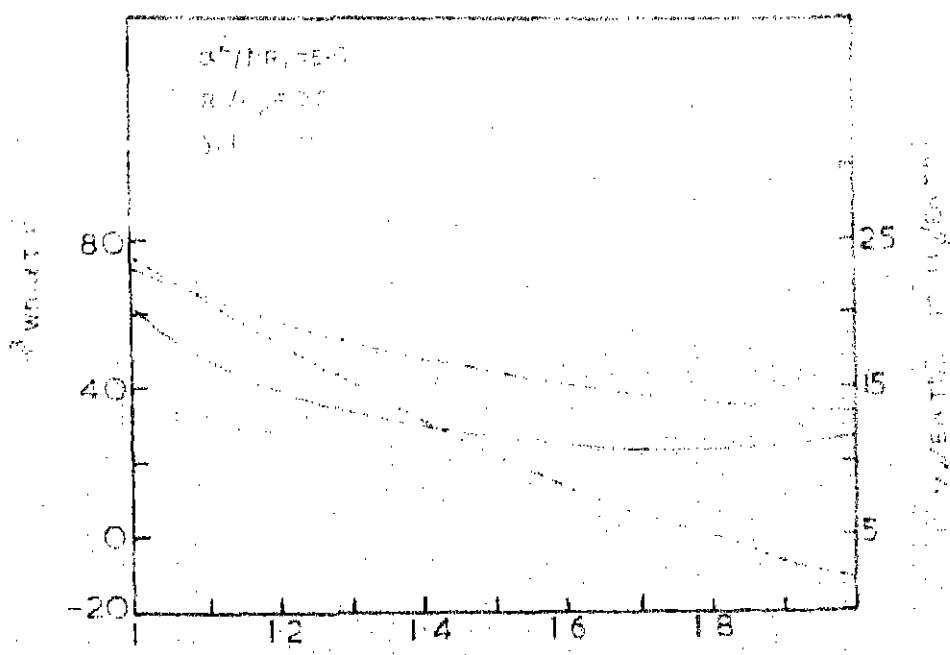
APPENDIX E

$\alpha^2/EI_B = 75 \times 10^6$

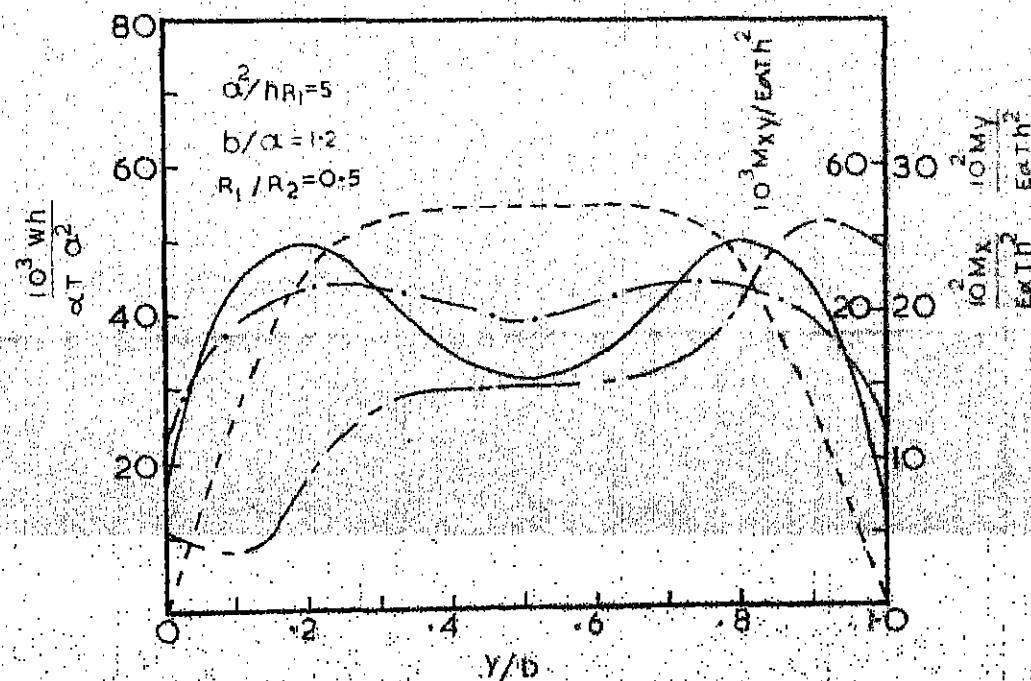
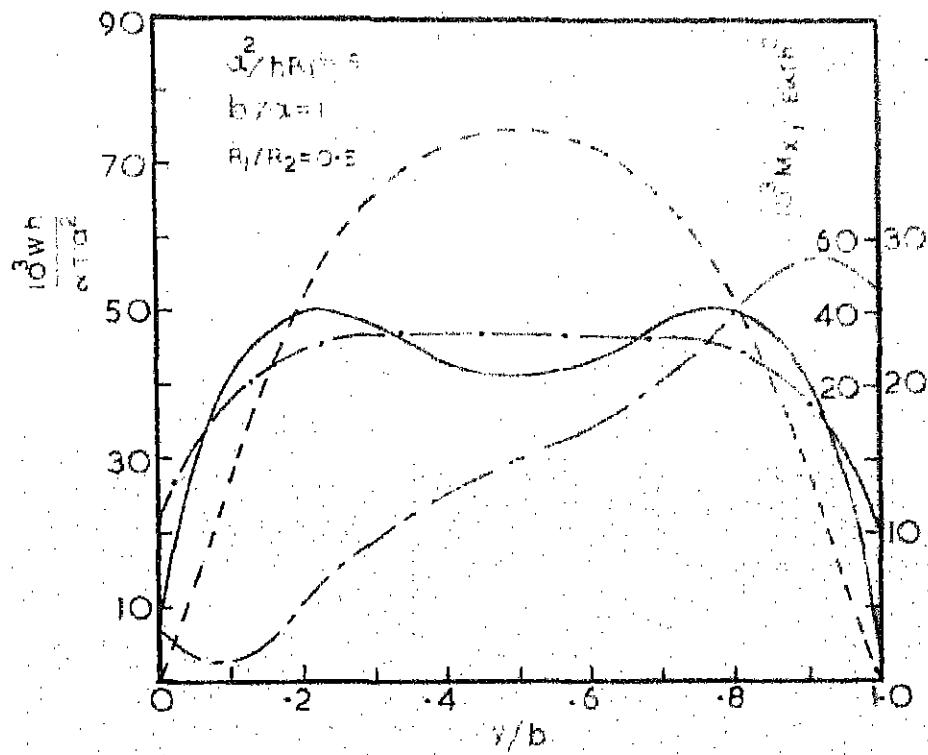
$b/a = 0.75$

$y/b = 0.5$

ALL EDGES SIMPLY SUPPORTED

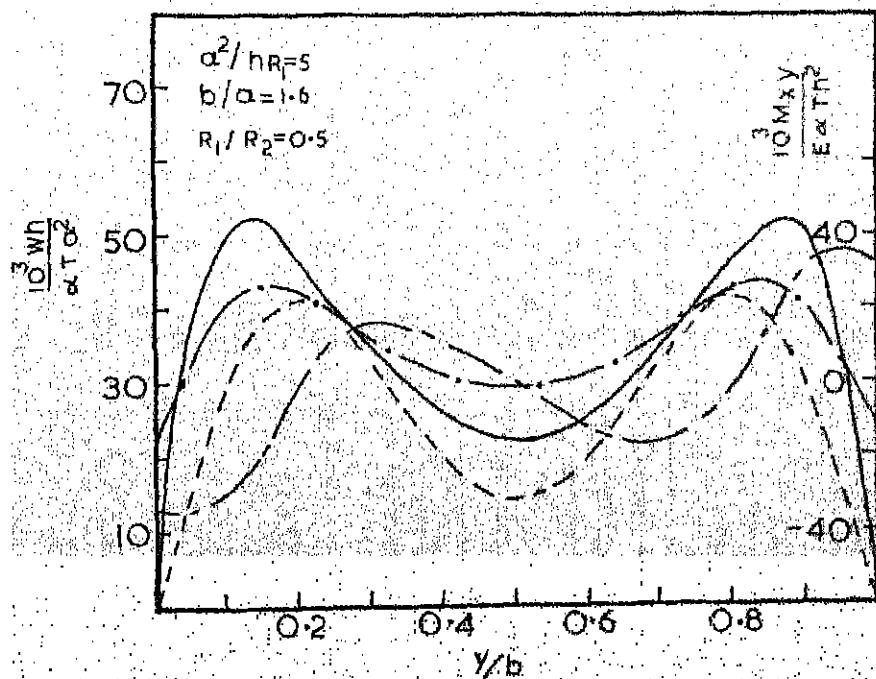
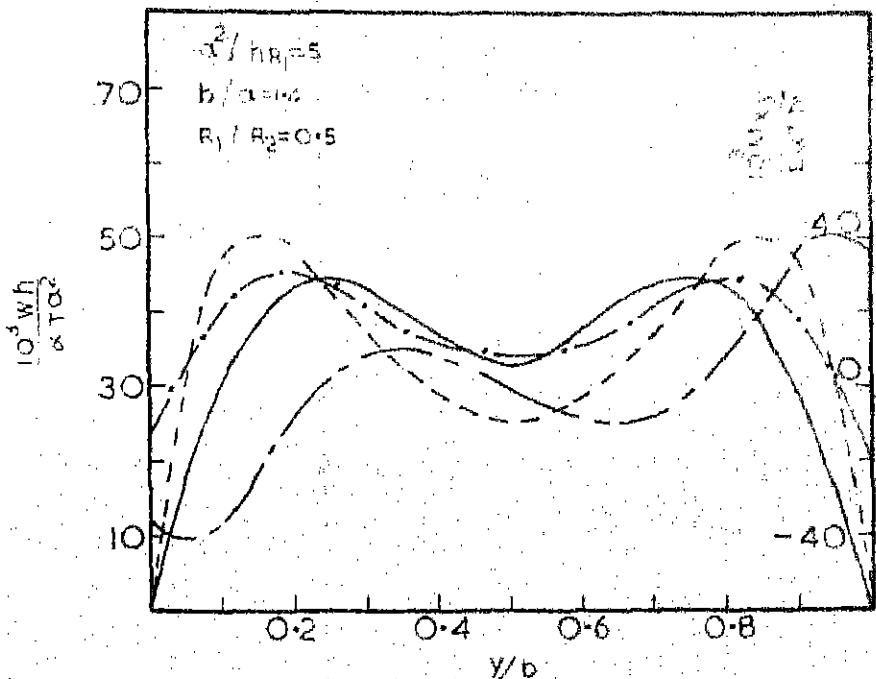


$10^3 \text{ Wh}/\alpha T \text{ cm}^2$
 $10^2 \text{ Myr}/\text{Earth}^2$ —— ALL EDGES SIMPLY
 $10^3 \text{ Mx}/\text{Earth}^2$ —— SUPPORTED
 $10^3 \text{ My}/\text{Earth}^2$ ——

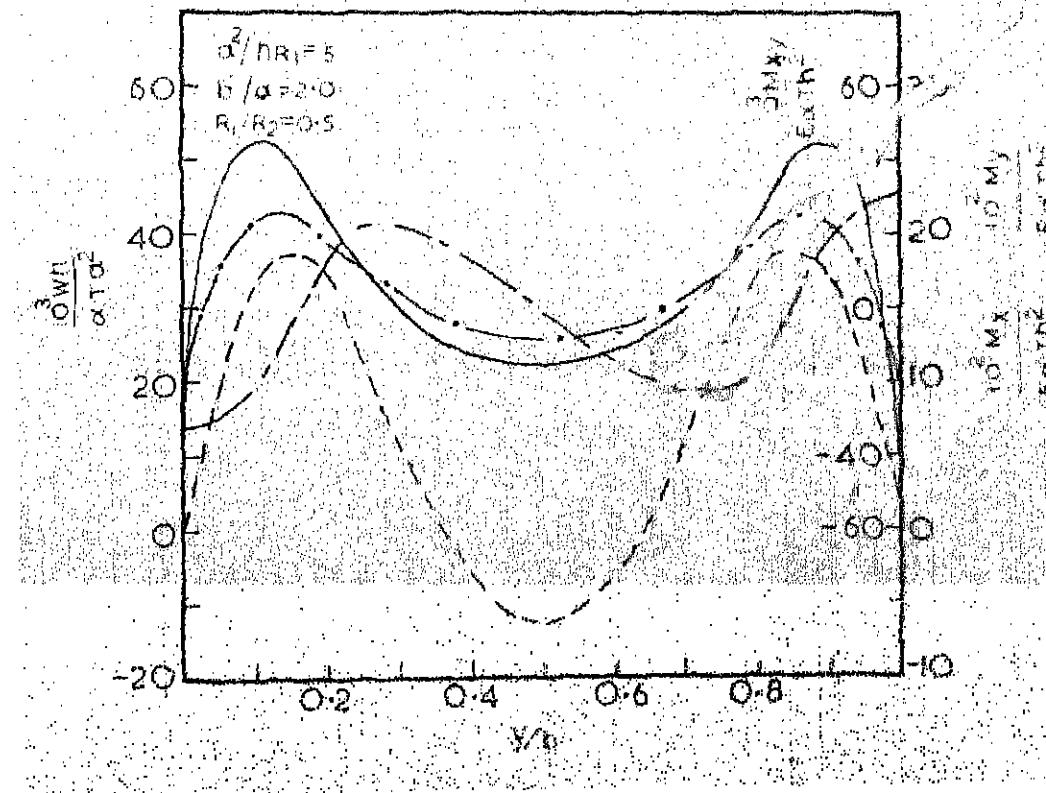
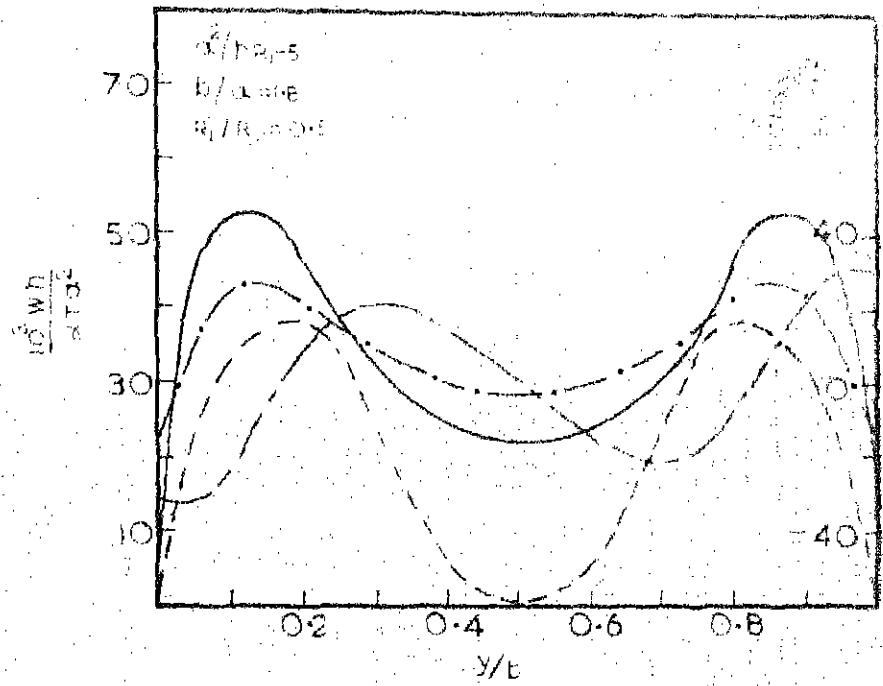


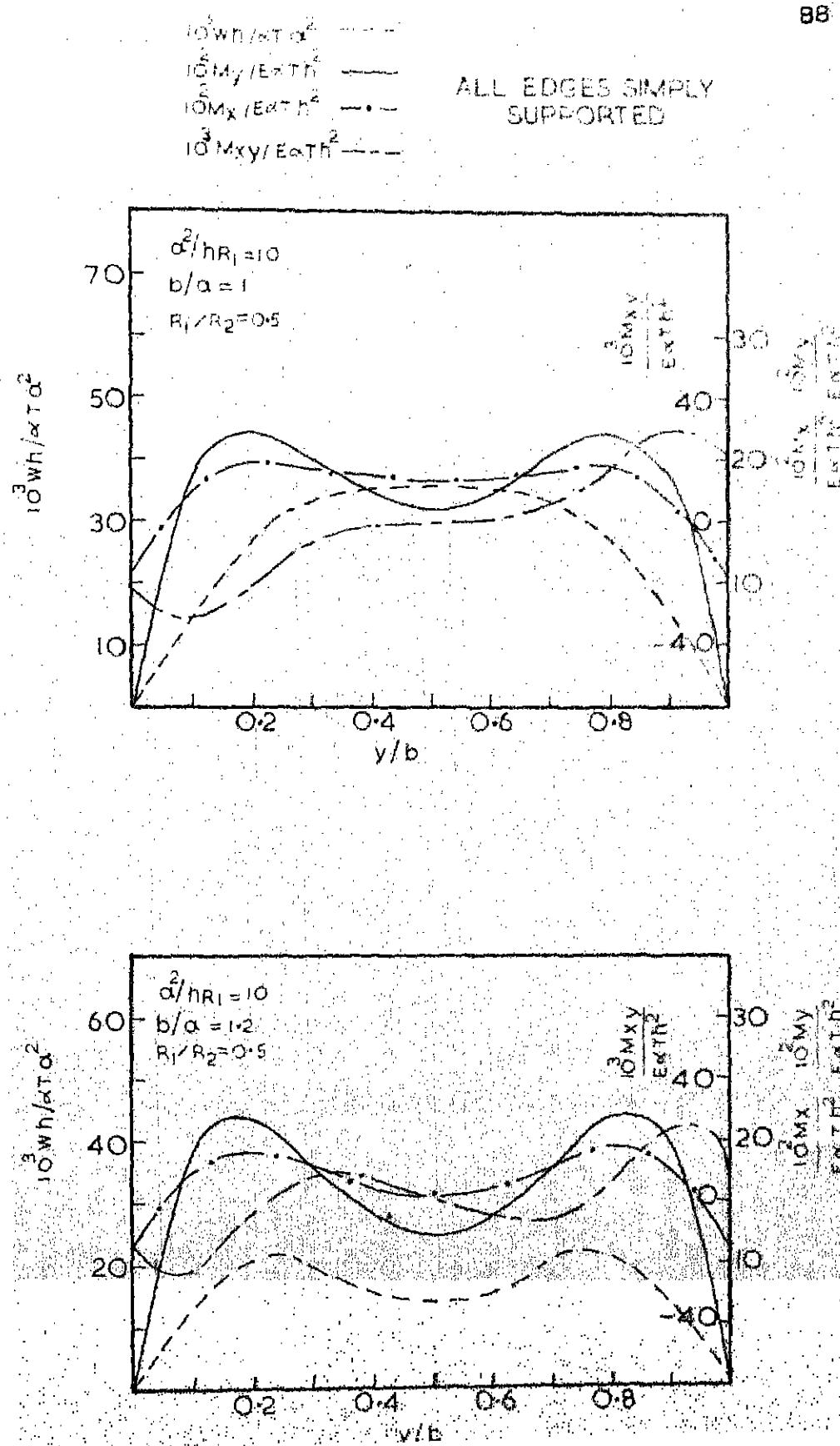
$10^3 \frac{Wn}{\alpha^2 T^2}$ ---
 $10^2 \frac{M_y}{E \alpha^2 T^2}$ ---
 $10^2 \frac{M_x}{E \alpha^2 T^2}$ ---
 $10^3 \frac{M_{xy}}{E \alpha^2 T^2}$ ---
 $\alpha^2 / h R_1 = 5$
 $b / a = 1.6$
 $R_1 / R_2 = 0.5$

ALL EDGES SIMPLY SUPPORTED



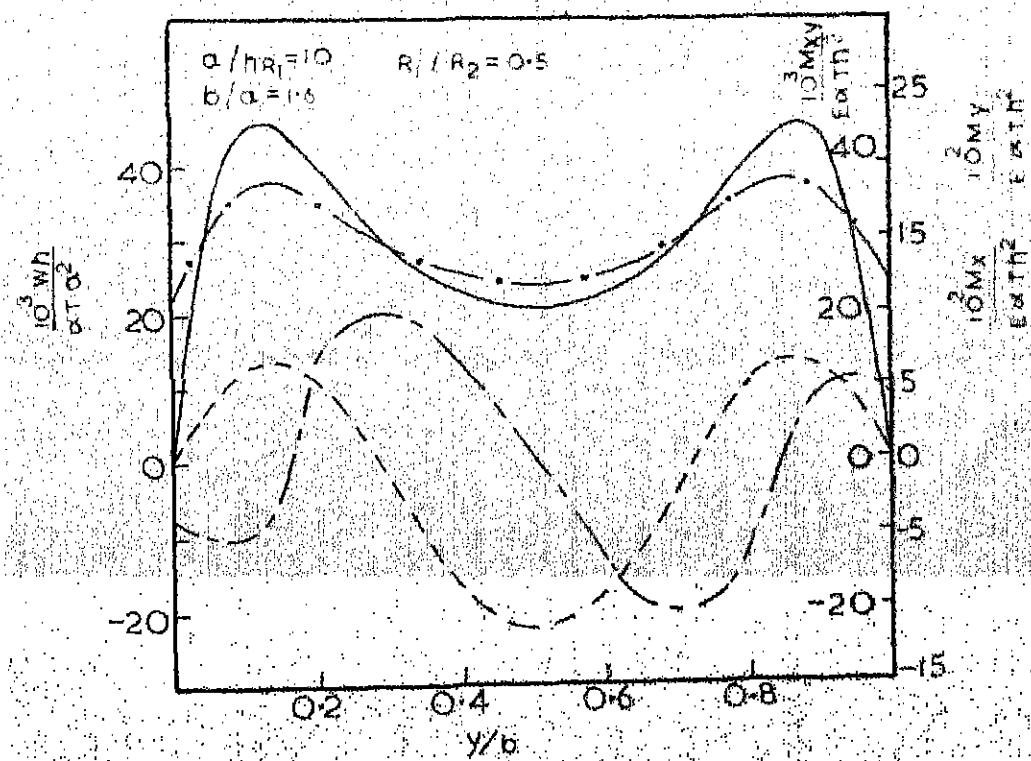
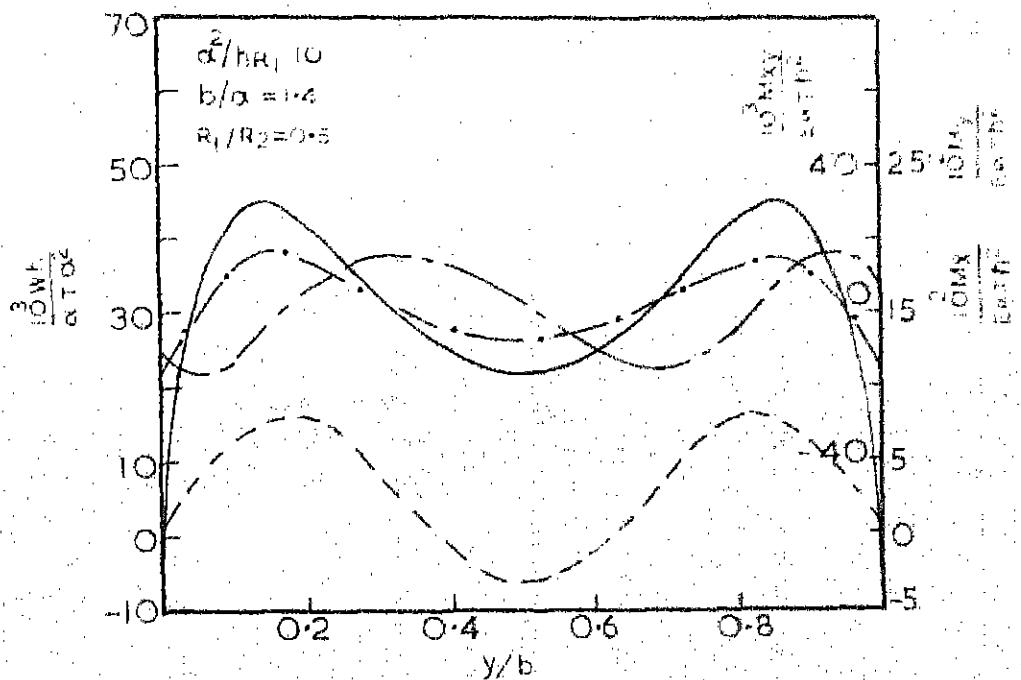
ALL SUPPORTED
BY MAX. STIFFNESS
SUPPORTED





10³ My/Earth

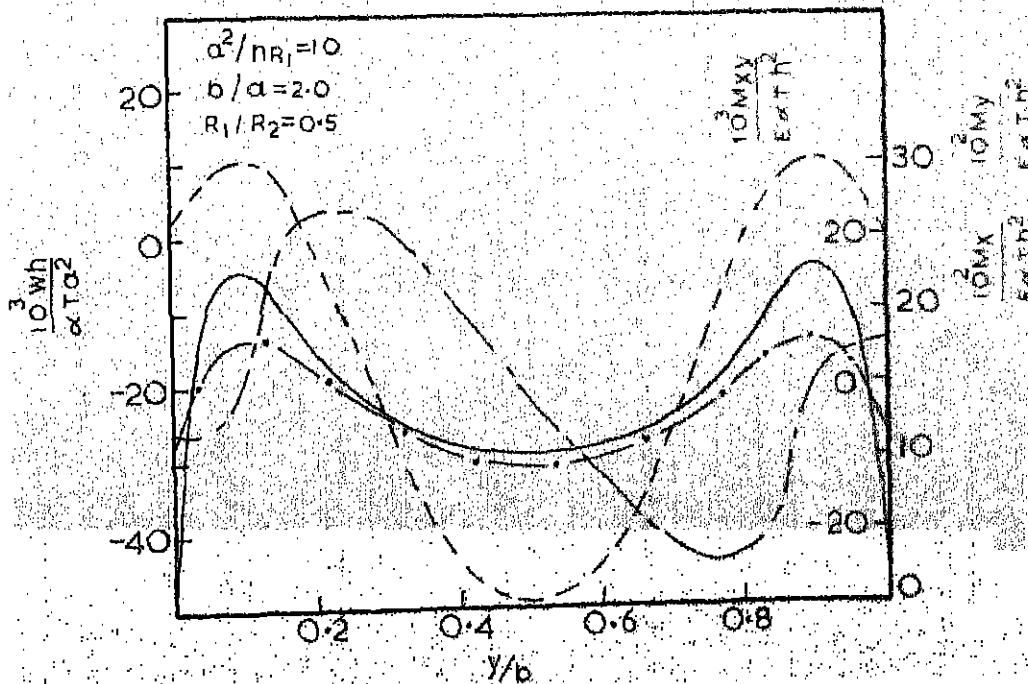
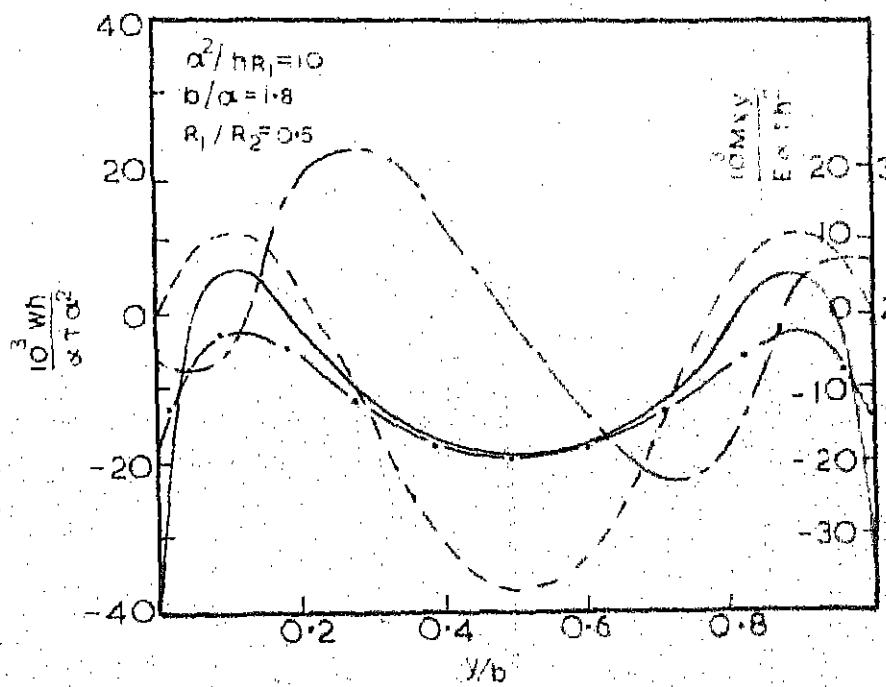
ALL EDGES SIMPLY SUPPORTED



$10^3 W/h \alpha^2$ ---
 $10^3 M_y/E \alpha^2 h^2$ ---
 $10^2 M_x/E \alpha^2 h^2$ ---
 $10^3 M_{xy}/E \alpha^2 h^2$ ---

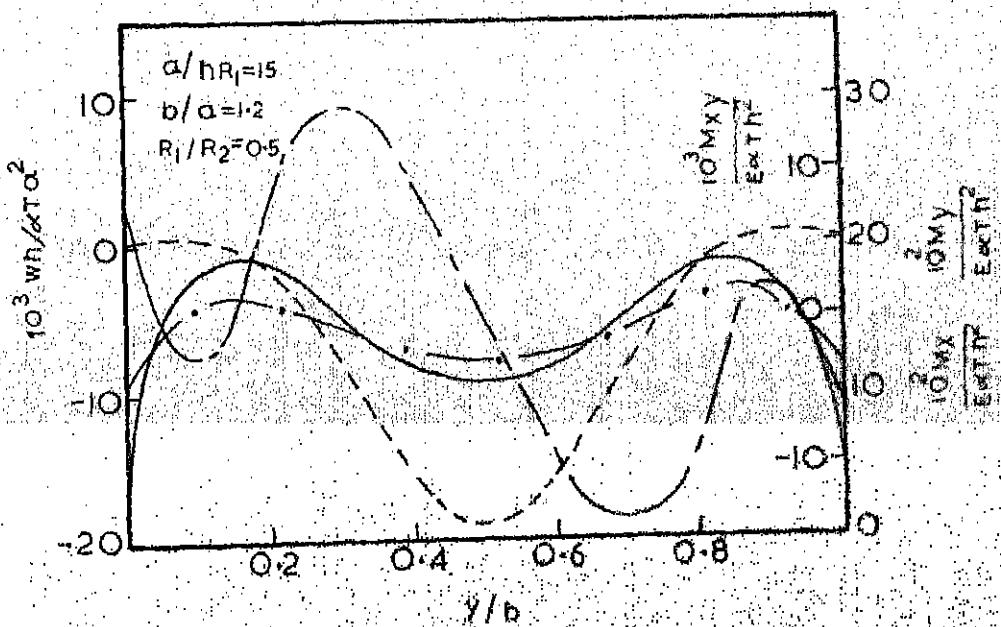
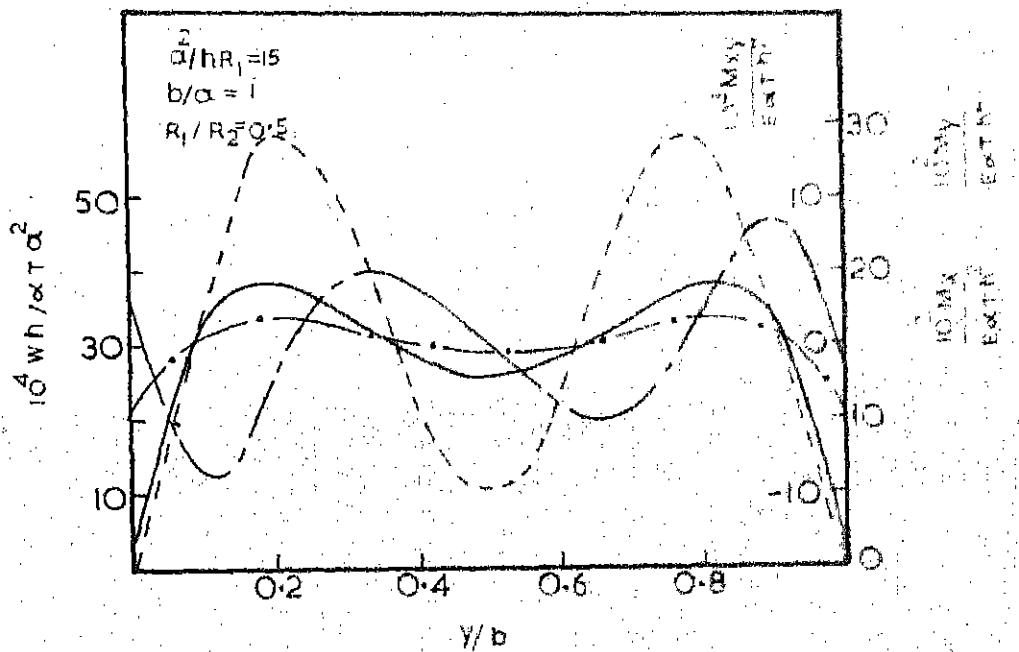
ALL EDGES SIMPLY SUPPORTED

90



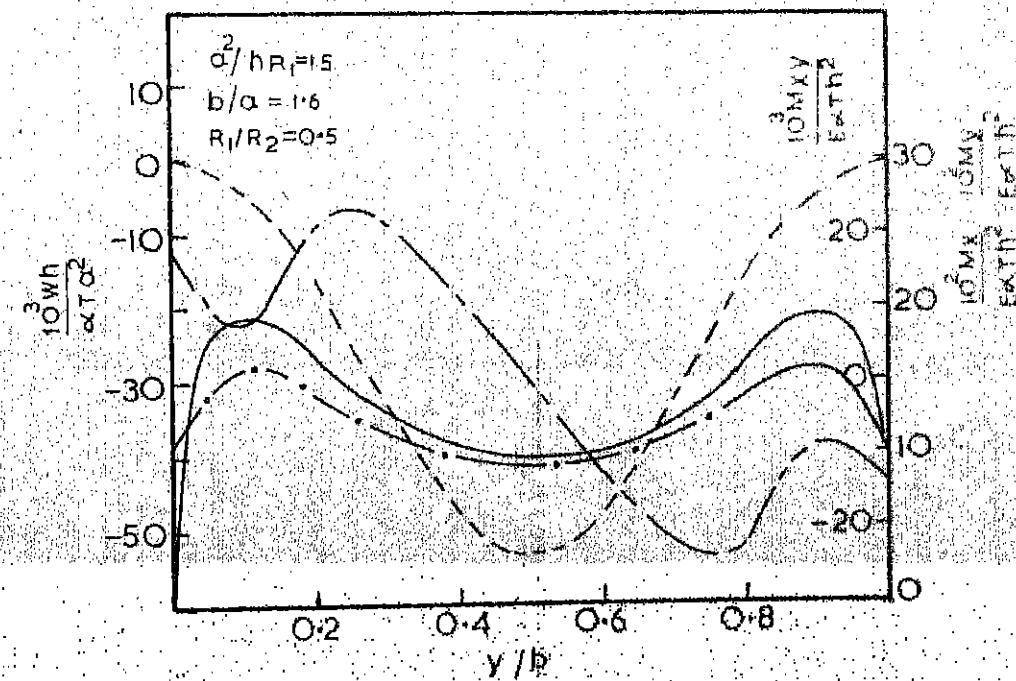
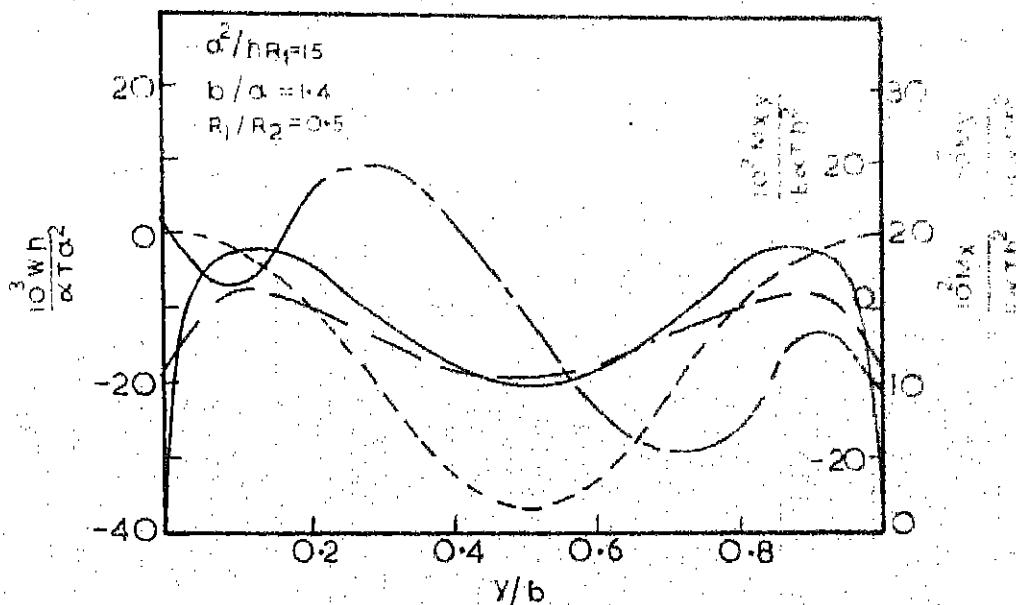
$10^3 wh/\alpha \tau \alpha^2$
 $10^3 My/E \alpha \tau h^2$
 $10^3 Mx/E \alpha \tau h^2$
 $10^3 Mxy/E \alpha \tau h^2$

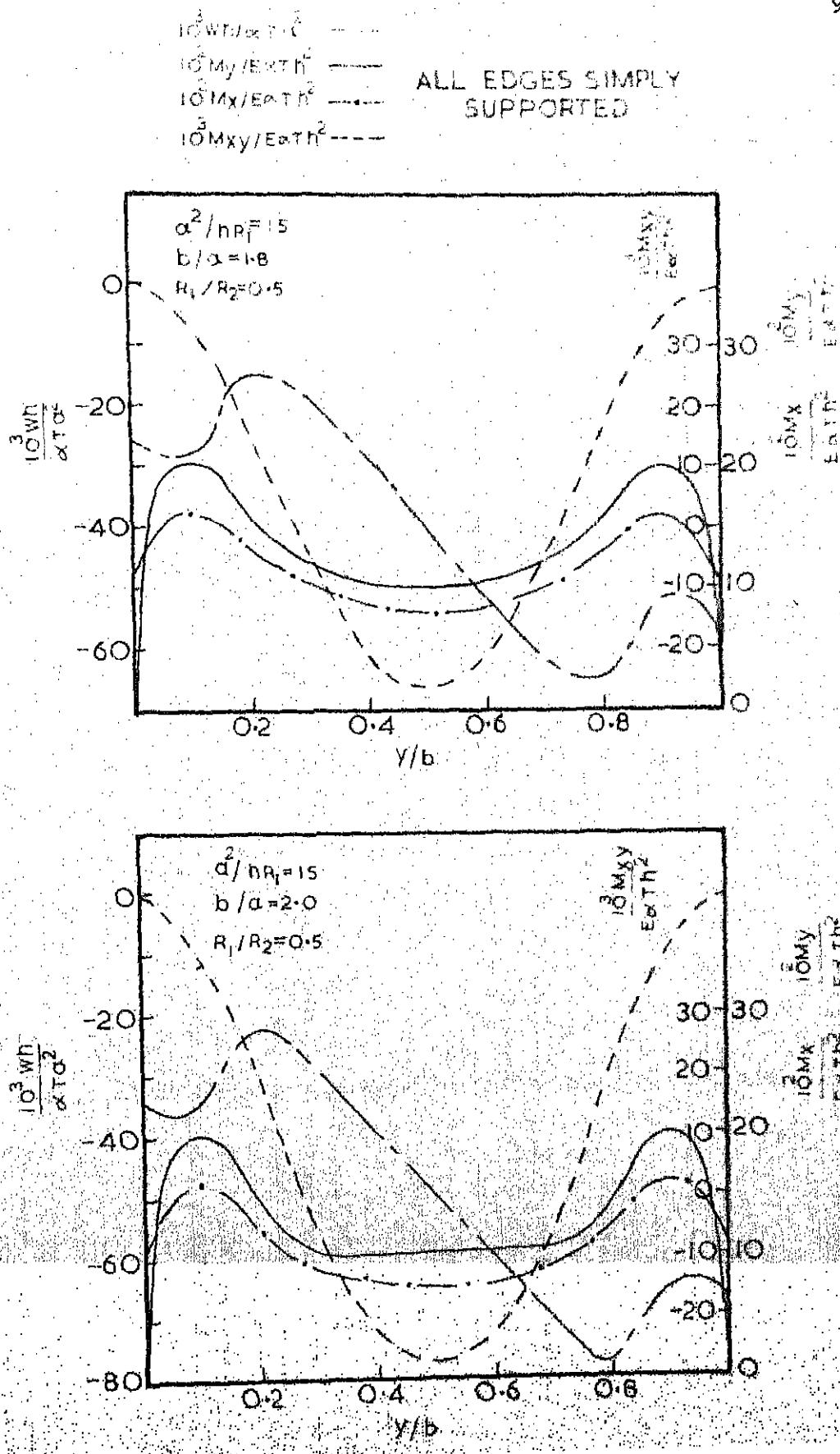
ALL EDGES SIMPLY SUPPORTED

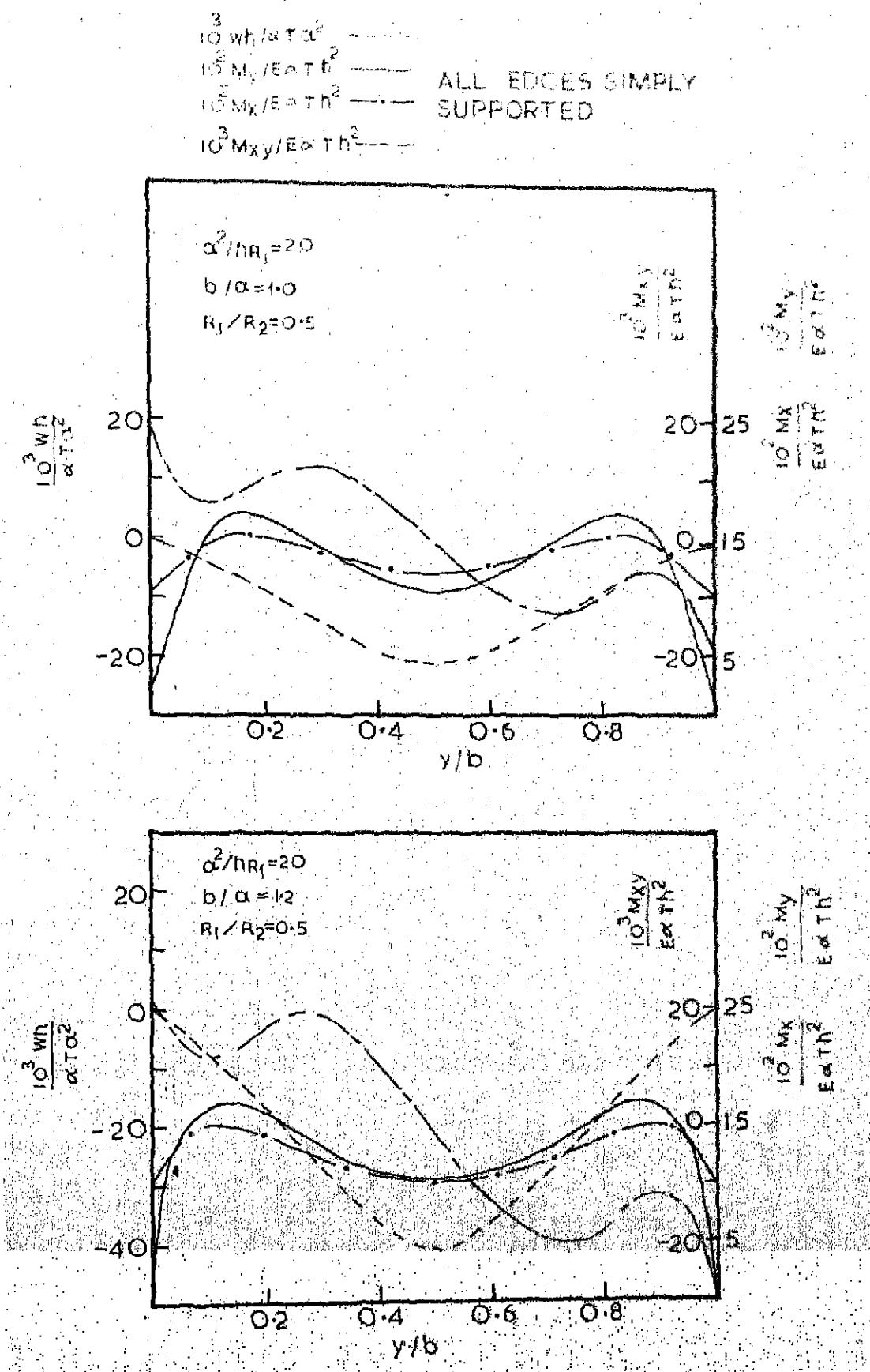


$10^3 W h / a^2$ — — —
 $10^3 M y / E a^2 h^2$ — — —
 $10^2 M x / E a^2 h^2$ — — —
 $10^3 M x y / E a^2 h^2$ — — —
 ALL EDGES SIMPLY
 SUPPORTED

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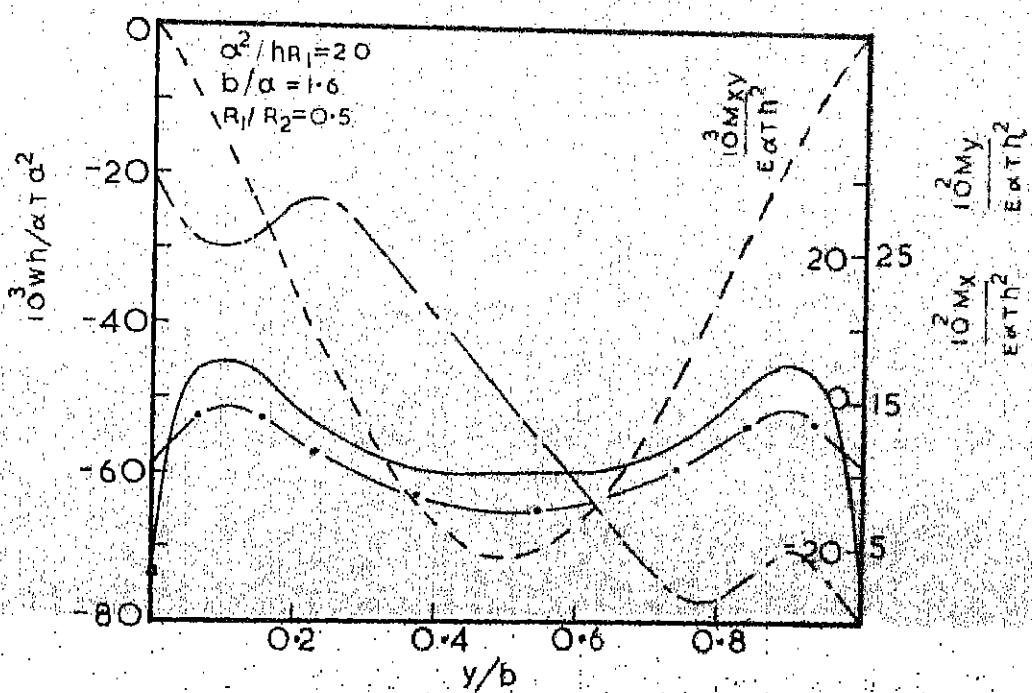
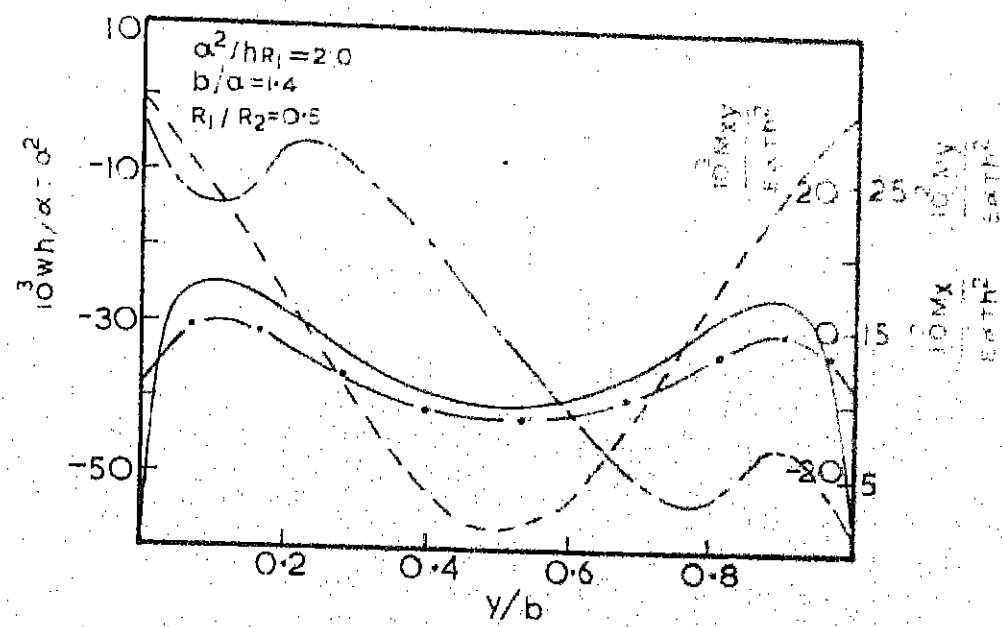


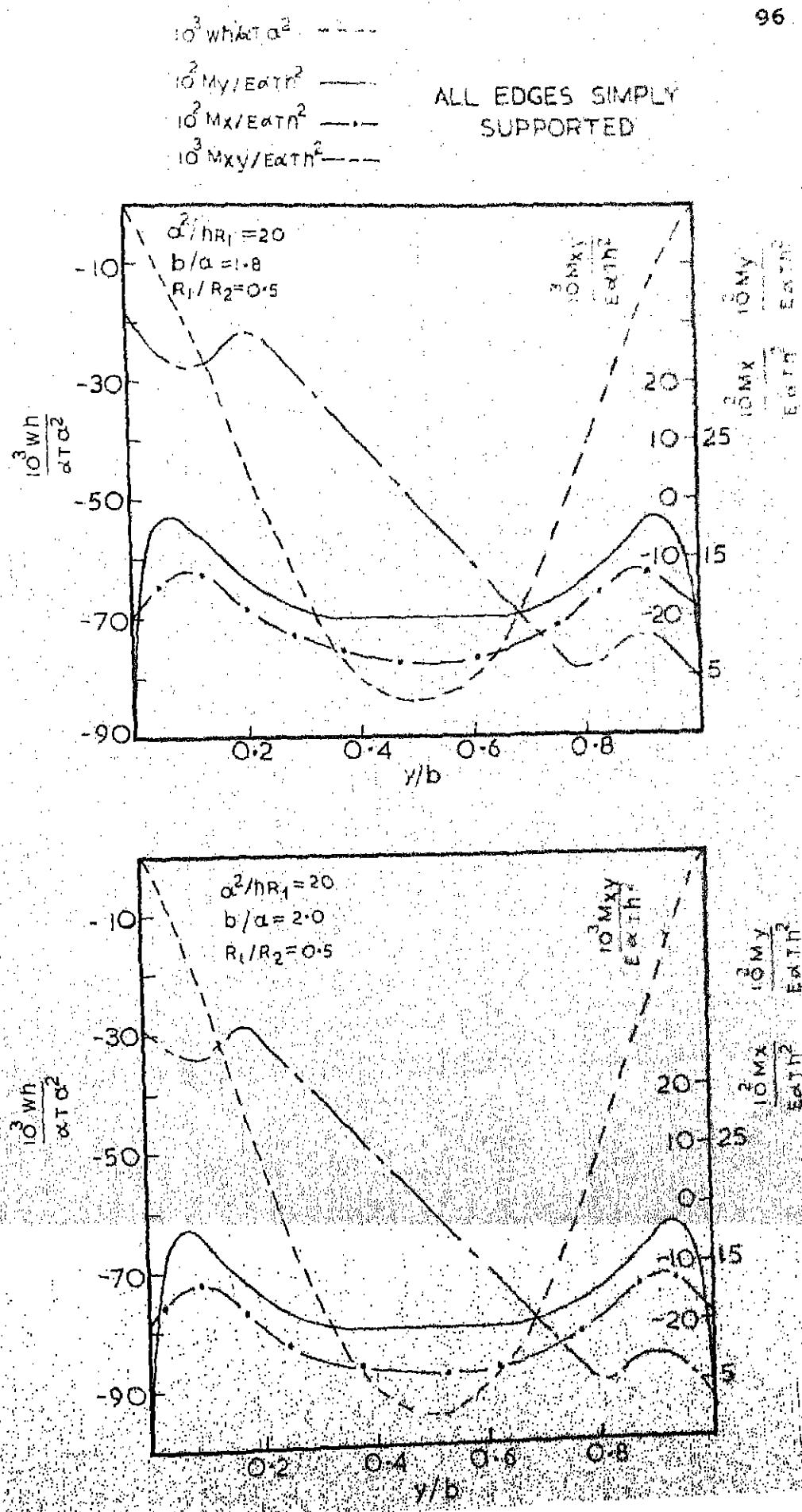




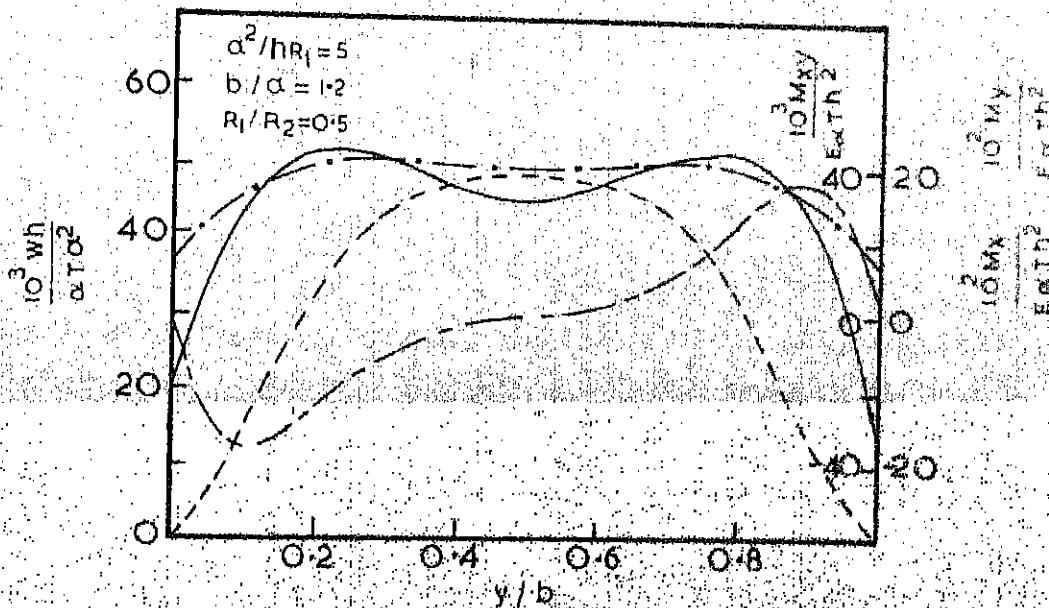
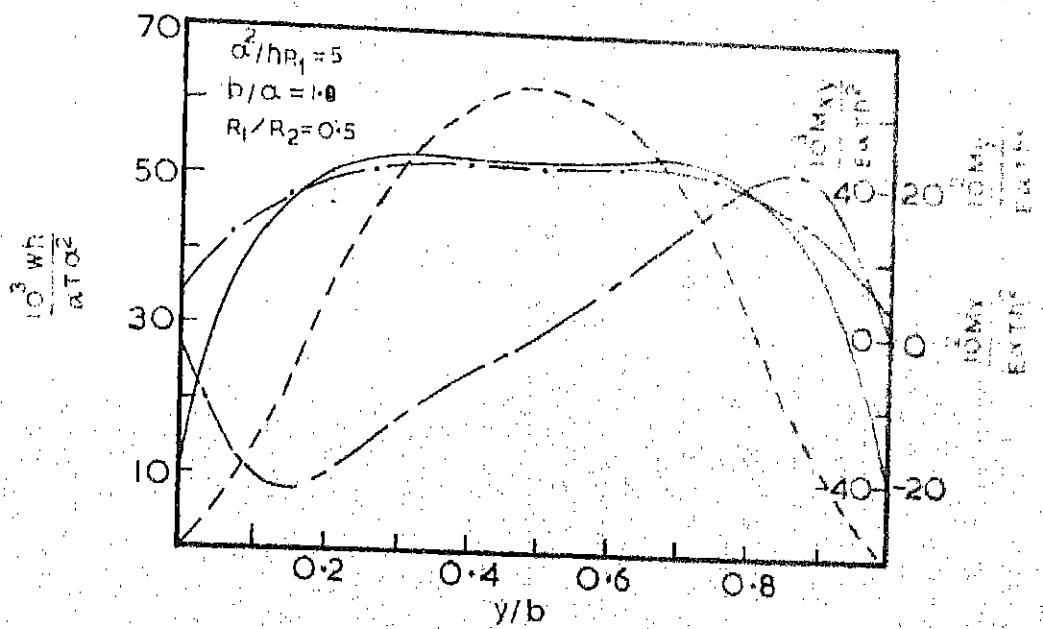
ALL EDGES SIMPLY
SUPPORTED

$10^3 wh/\alpha T \alpha^2$ ---
 $10^2 My/E \alpha T h^2$ ---
 $10^2 Mx/E \alpha T h^2$ ---
 $10^3 Mxy/E \alpha T h^2$ ---

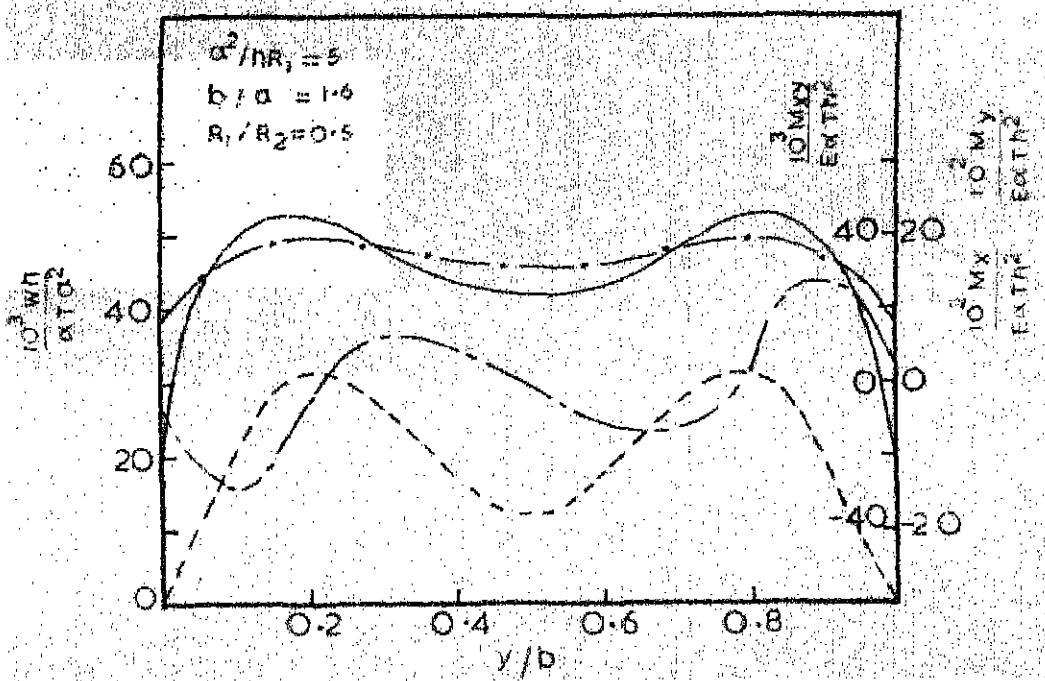
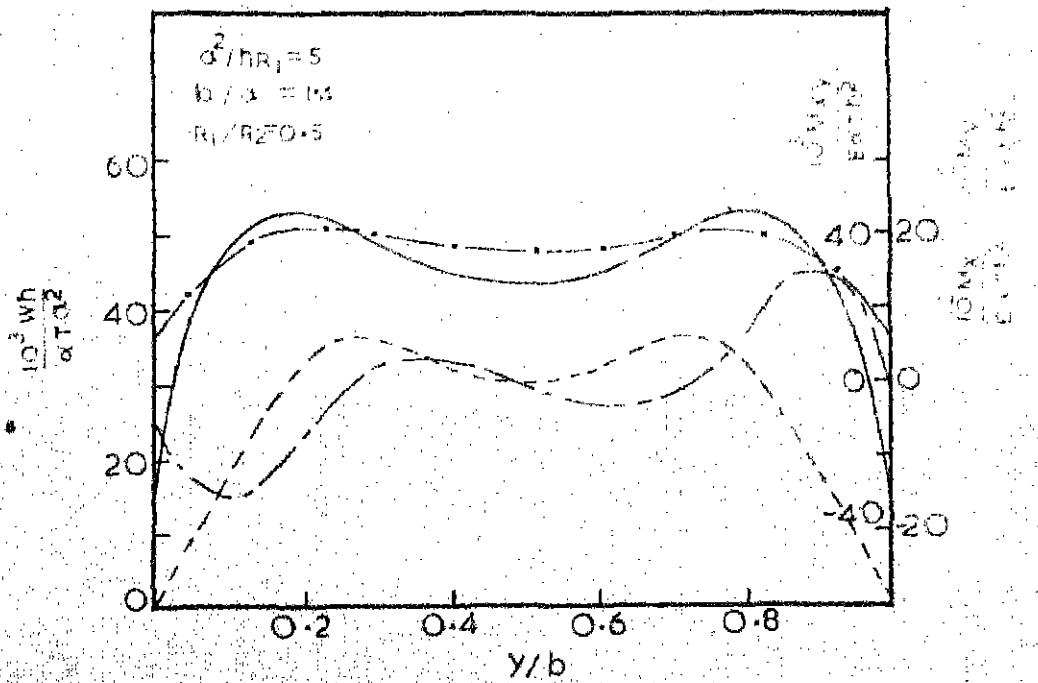




$10^3 \frac{wh}{\alpha^2}$ --- TWO EDGES CLAMPED
 $10^6 \frac{My}{E \alpha^2 h^2}$ --- TWO EDGES SIMPLY SUPPORTED
 $10^2 \frac{Mx}{E \alpha^2 h^2}$ ---
 $10^3 \frac{Mxy}{E \alpha^2 h^2}$ ---
 $\alpha^2/hR_1 = 5$
 $b/a = 1.0$
 $R_1/R_2 = 0.5$

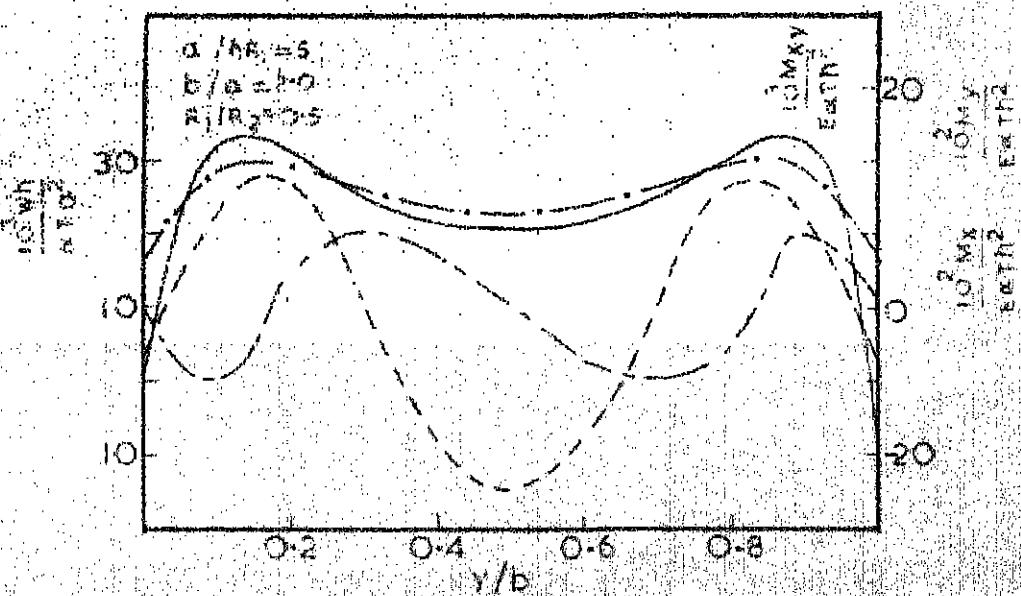
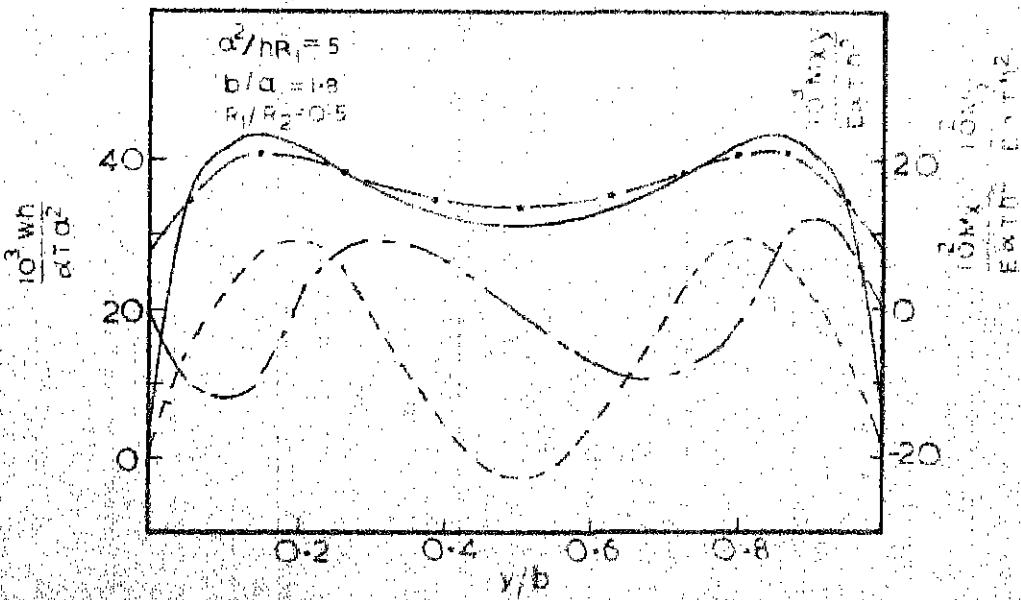


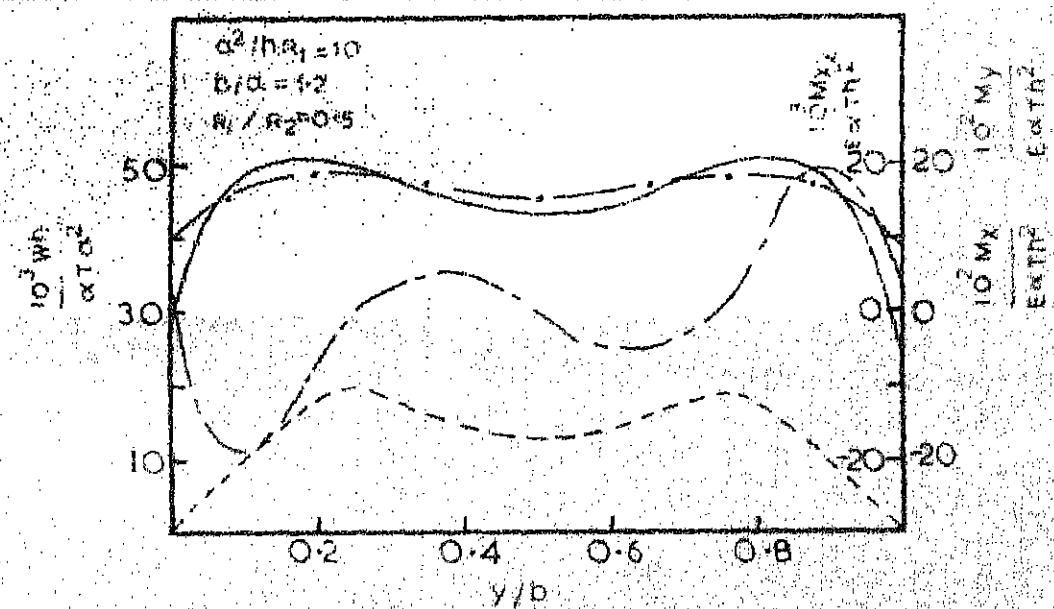
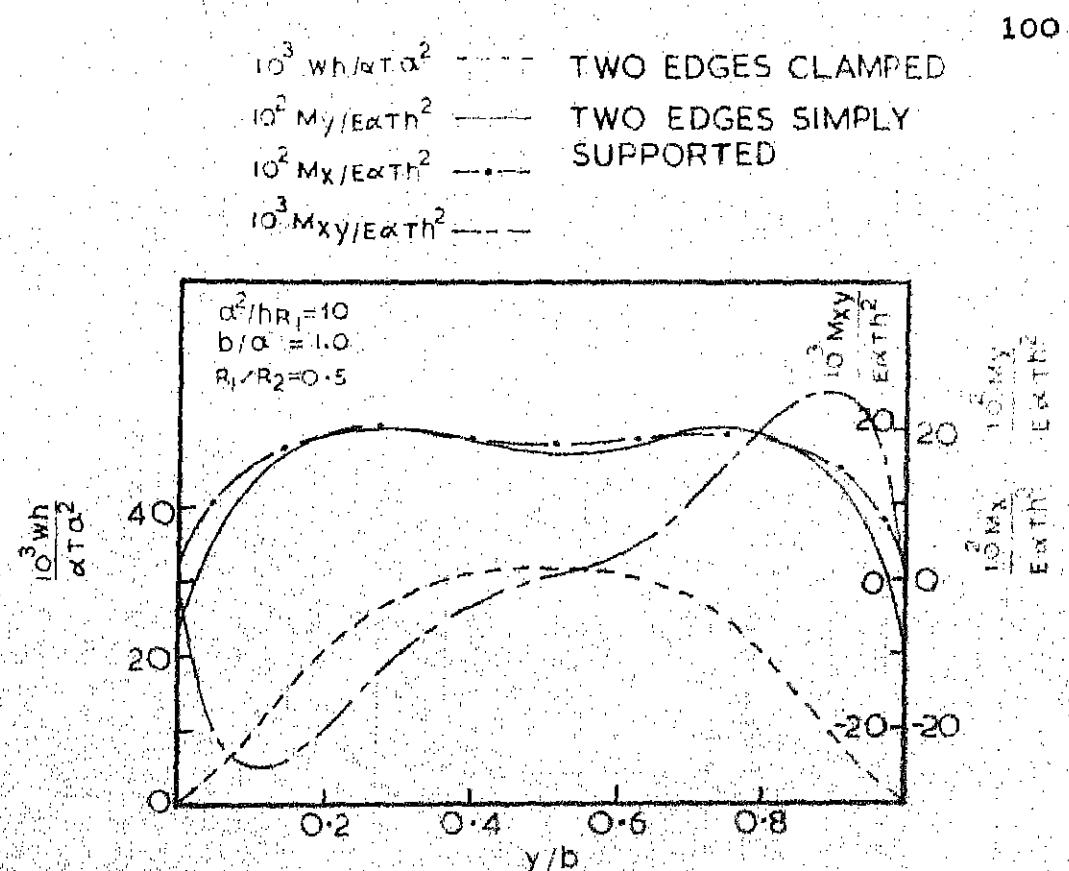
$10^3 \frac{W^3}{h^2 \alpha^2}$ - - - - - TWO EDGES CLAMPED
 $10^3 \frac{M_y}{E \alpha^2 T^2}$ - - - - - TWO EDGES SIMPLY SUPPORTED
 $10^3 \frac{M_x}{E \alpha^2 T^2}$ - - - - -
 $10^3 \frac{M_{xy}}{E \alpha^2 h^2 T^2}$ - - - - -



$10^3 W^3 \rho T a^2$ —
 $10^3 M_y / E a T h^2$ —
 $10^2 M_x / E a T h^2$ —
 $10^3 M_{xy} / E a T h^2$ —

TWO EDGES CLAMPED
 TWO EDGES SIMPLY
 SUPPORTED



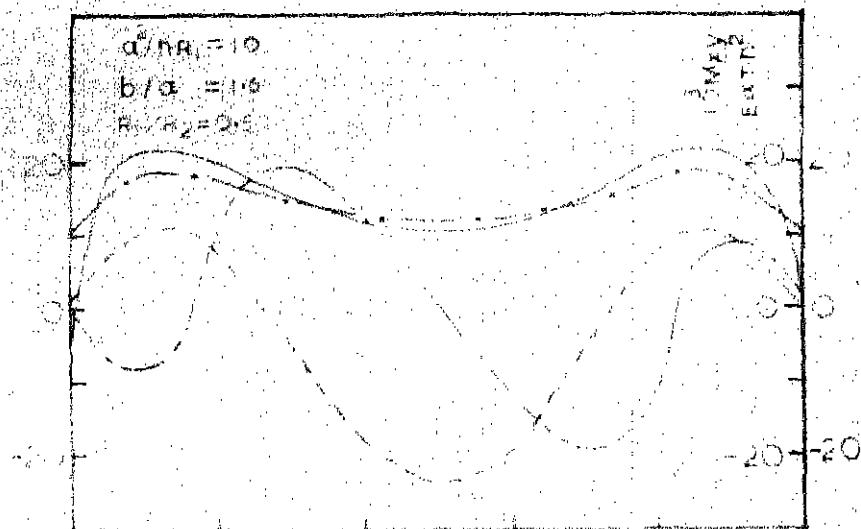
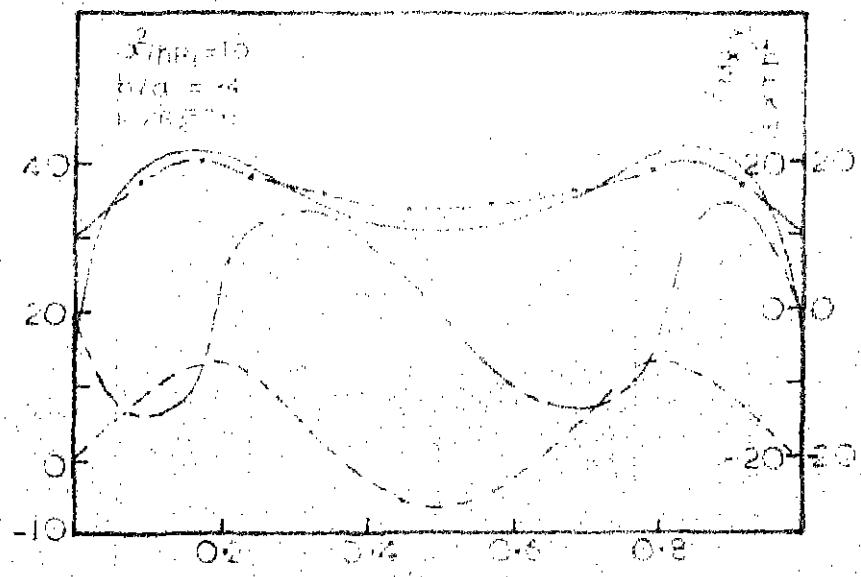


RESULTS OF THE COMPUTATION OF THE STABILITY OF THE AIRFOIL

FOR THE CASE OF TWO PLATES, SIMILARLY SHAPED, BUT

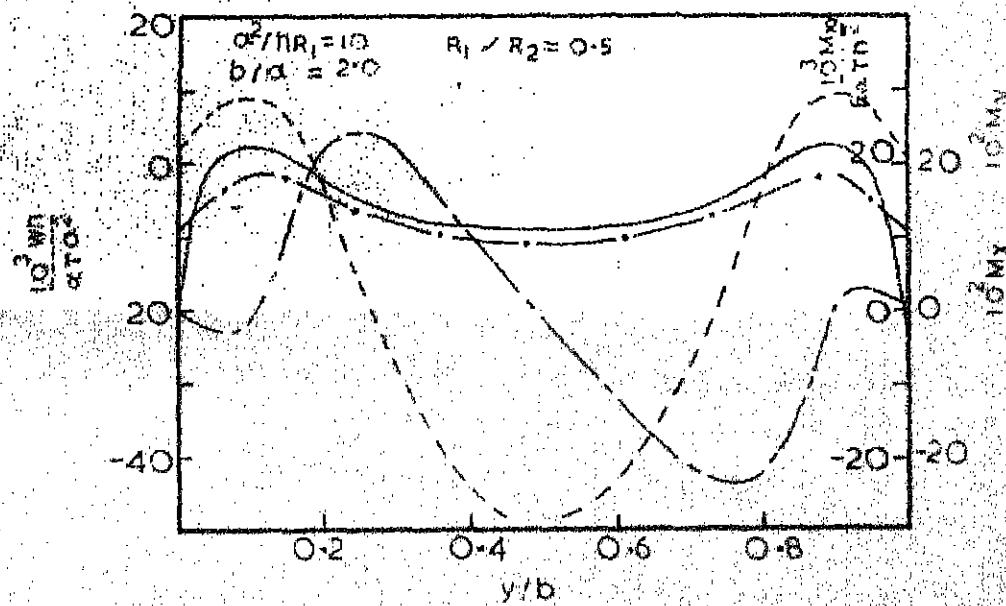
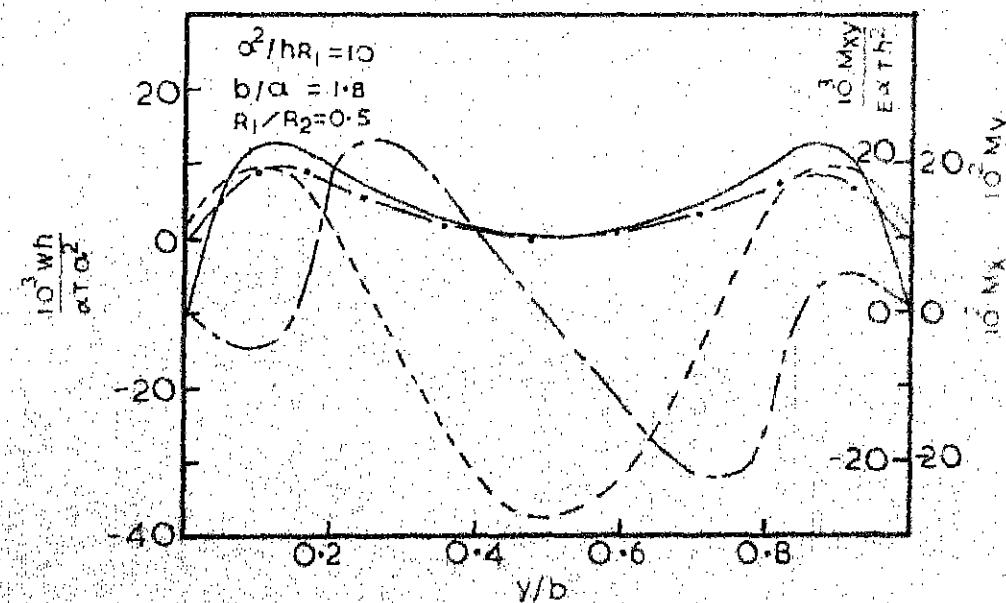
FOR $M_{\infty, A} = 1.2$ AND

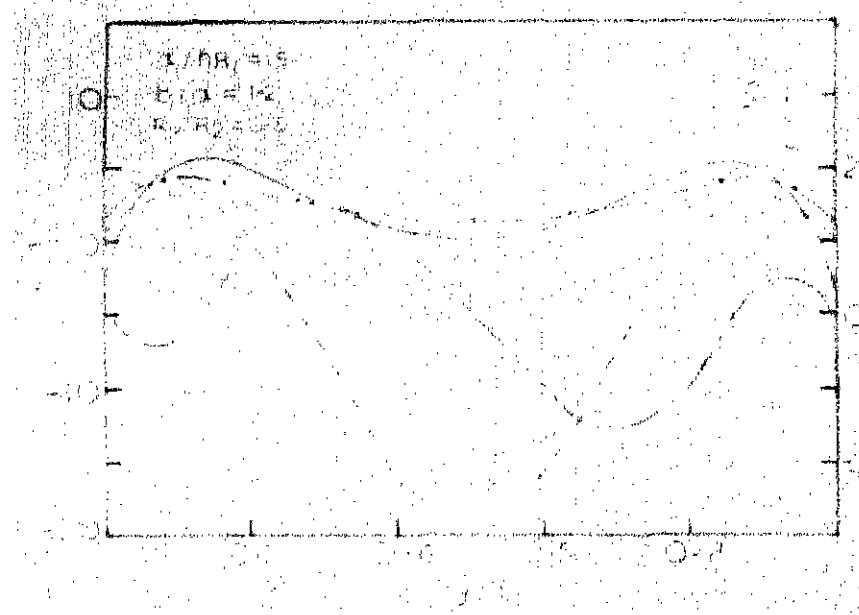
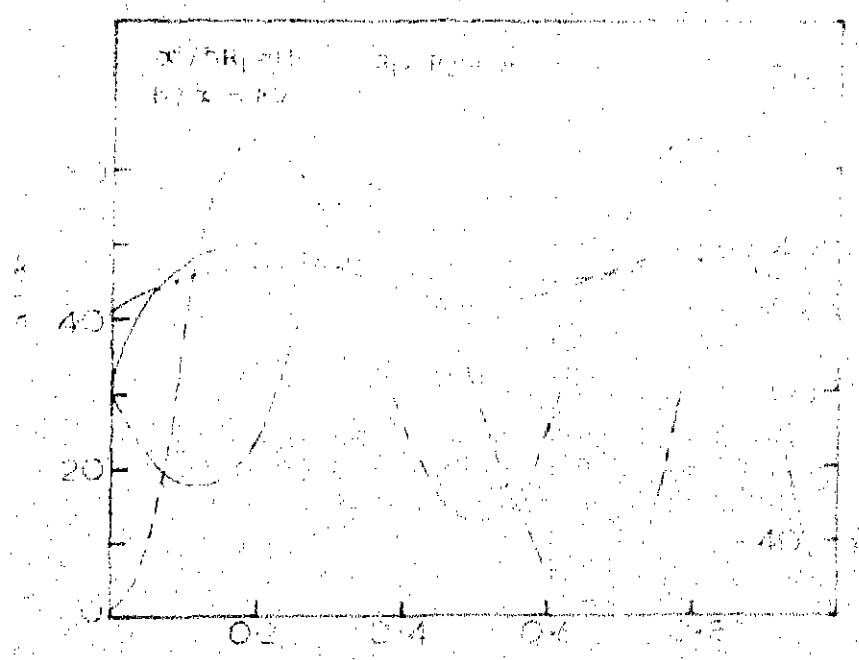
FOR $M_{\infty, B} = 1.2$



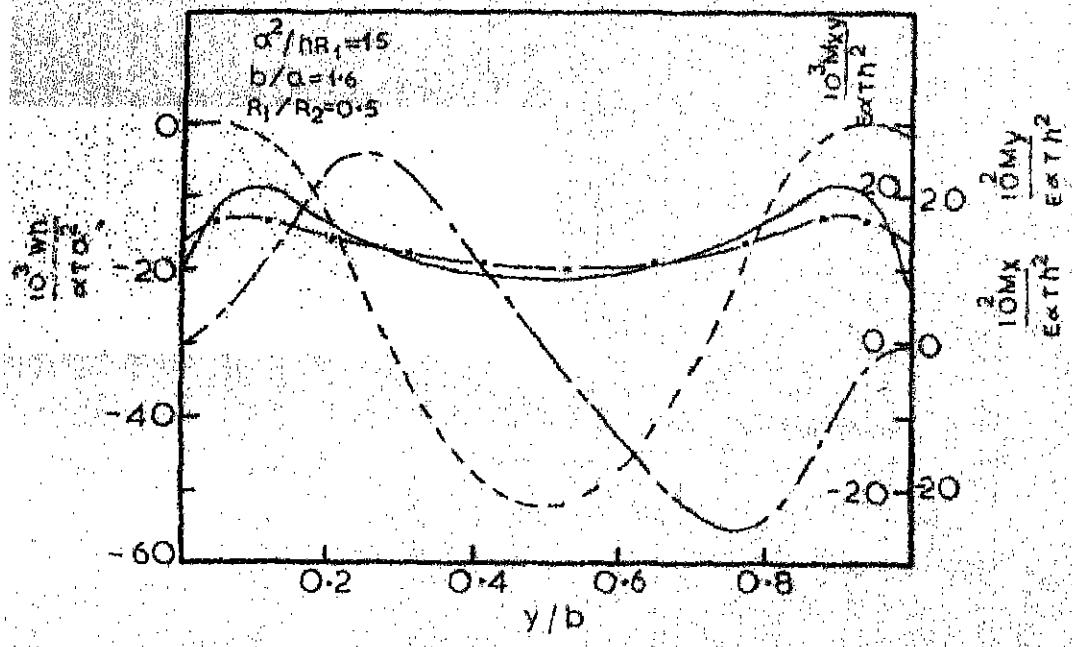
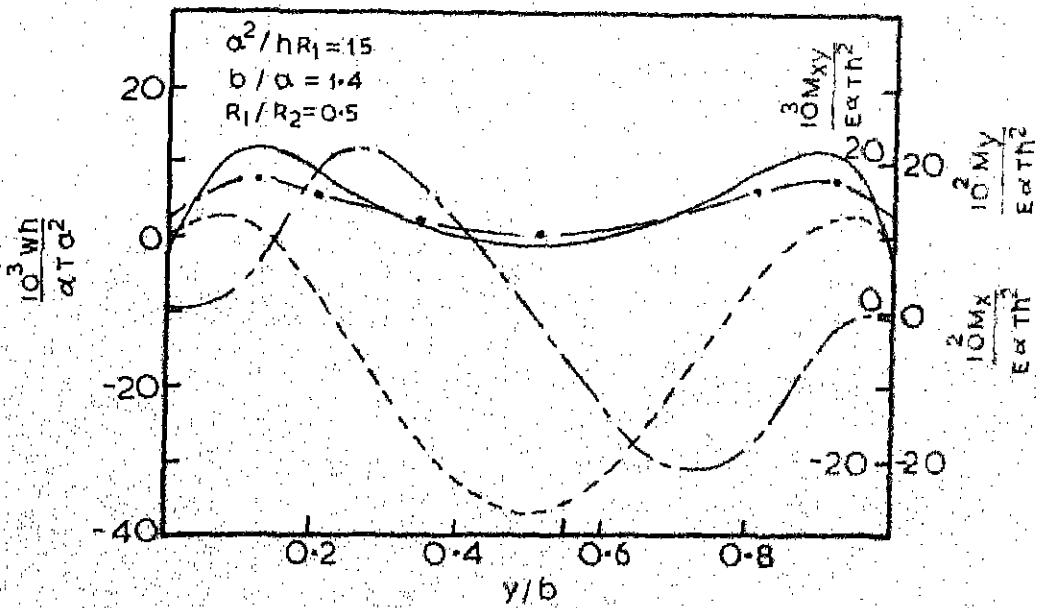
$\frac{10^3 Wh}{\alpha T^2}$ ---
 $\frac{10^2 My}{E a^2 h^2}$ ---
 $\frac{10^2 Mx}{E a^2 h^2}$ ---
 $\frac{10^3 Mxy}{E a^2 h^2}$ ---

TWO EDGES CLAMPED
 TWO EDGES SIMPLY
 SUPPORTED

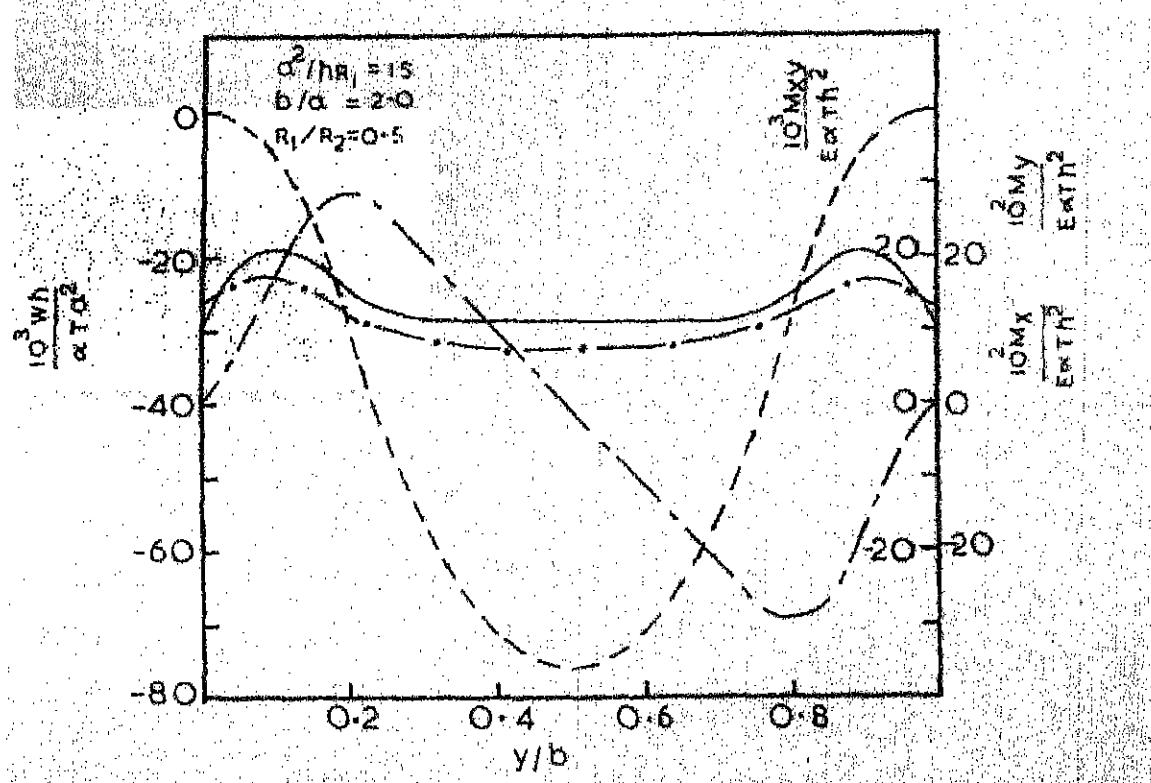
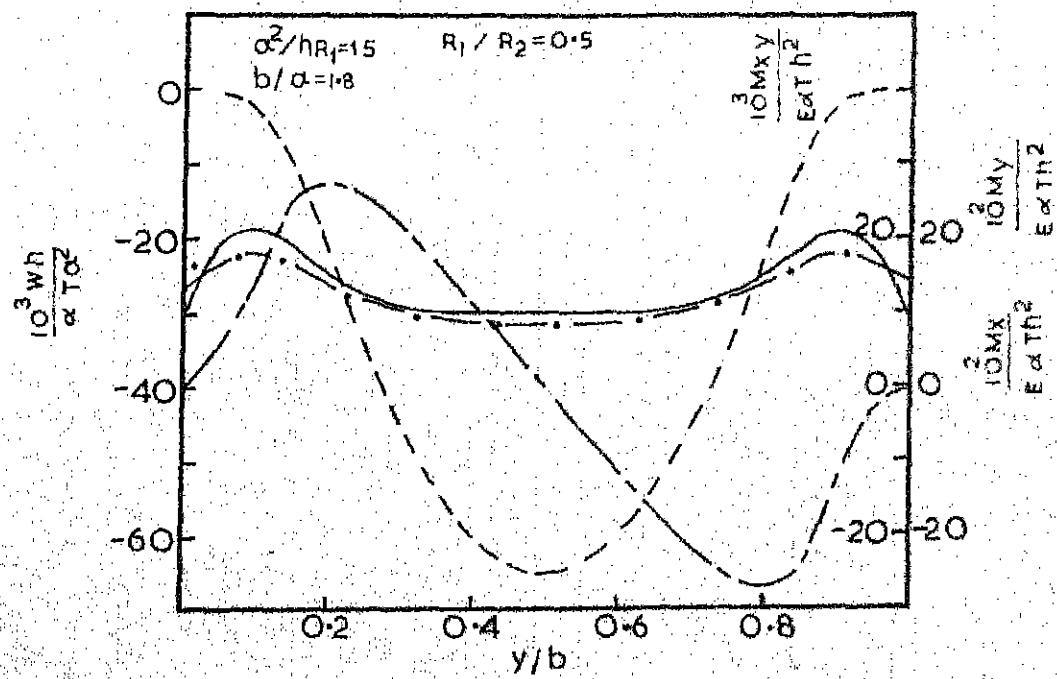




$10^3 \text{ wh}/\alpha^2$ - - - - - TWO EDGES CLAMPED
 $10^2 \text{ My}/\text{EaTH}^2$ - - - - - TWO EDGES SIMPLY
 $10^2 \text{ Mx}/\text{EaTH}^2$ - - - - SUPPORTED
 $10^3 \text{ Mxy}/\text{EaTH}^2$ - - - - -

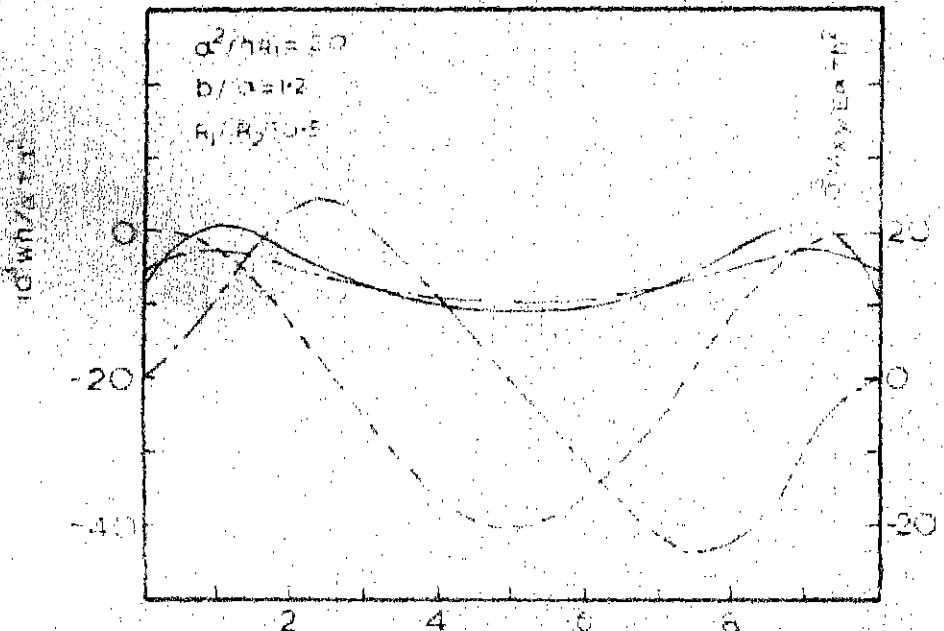
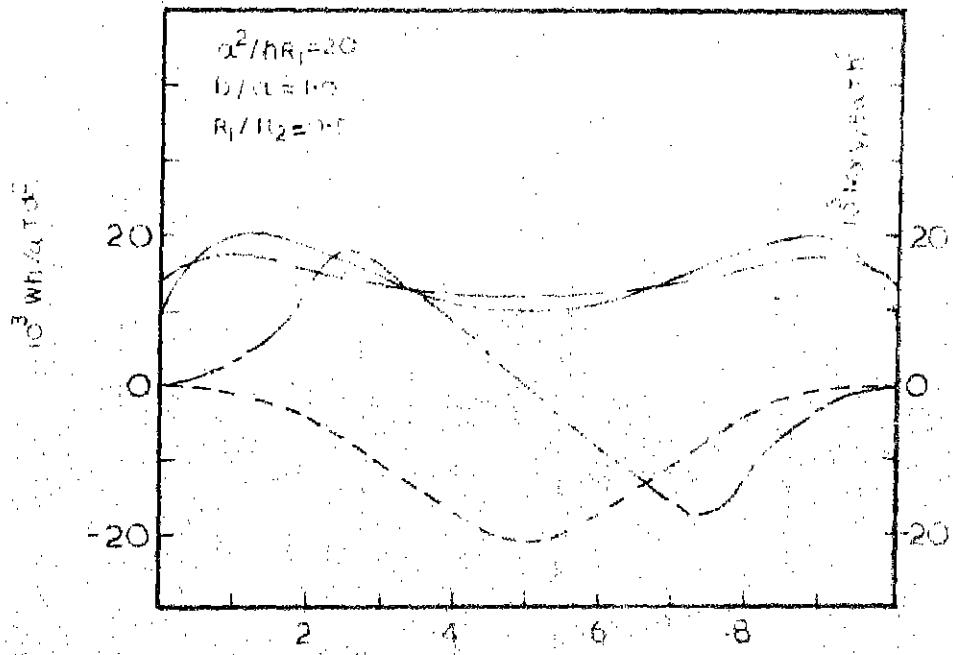


$10^3 \frac{wh}{\alpha T \alpha^2}$ - - - TWO EDGES CLAMPED
 $10^2 \frac{My}{E \alpha Th^2}$ ——— TWO EDGES SIMPLY
 $10^2 \frac{Mx}{E \alpha Th^2}$ - - - SUPPORTED
 $10^3 \frac{Mxy}{E \alpha Th^2}$ - - - - -



$\sigma^2/\alpha = 0.0001$
 $10M_y/E\alpha\theta^2 = 0.0001$
 $10M_x/E\alpha\theta^2 = 0.0001$
 $10K_y/E\alpha\theta^2 = 0.0001$

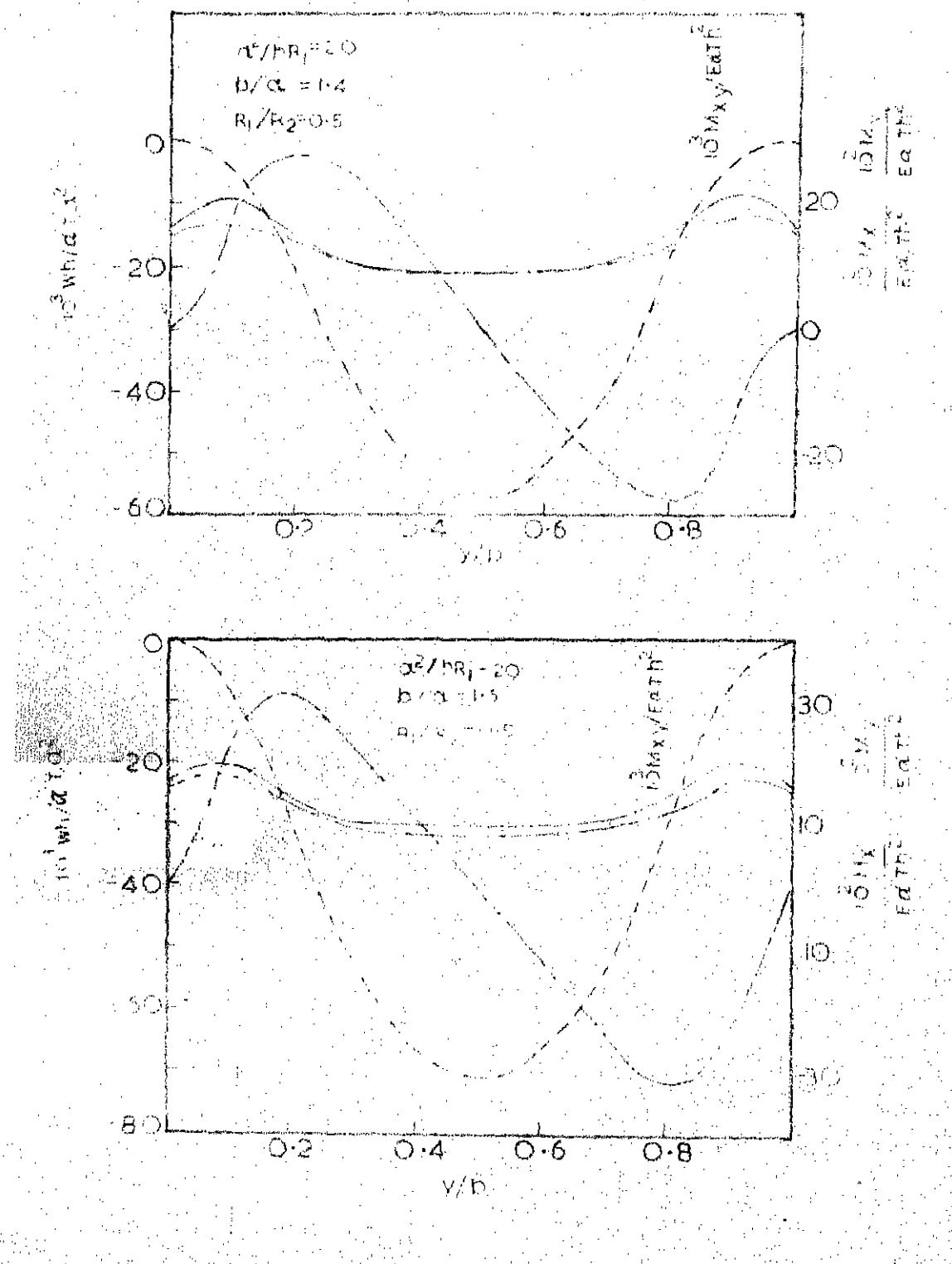
TWO EDGES SIMPLY SUPPORTED
 TWO EDGES CLAMPED



$\alpha^3 \text{ white } \text{EARTH}$
 10^3 My/Earth
 10^3 My/Earth
 10^3 My/Earth

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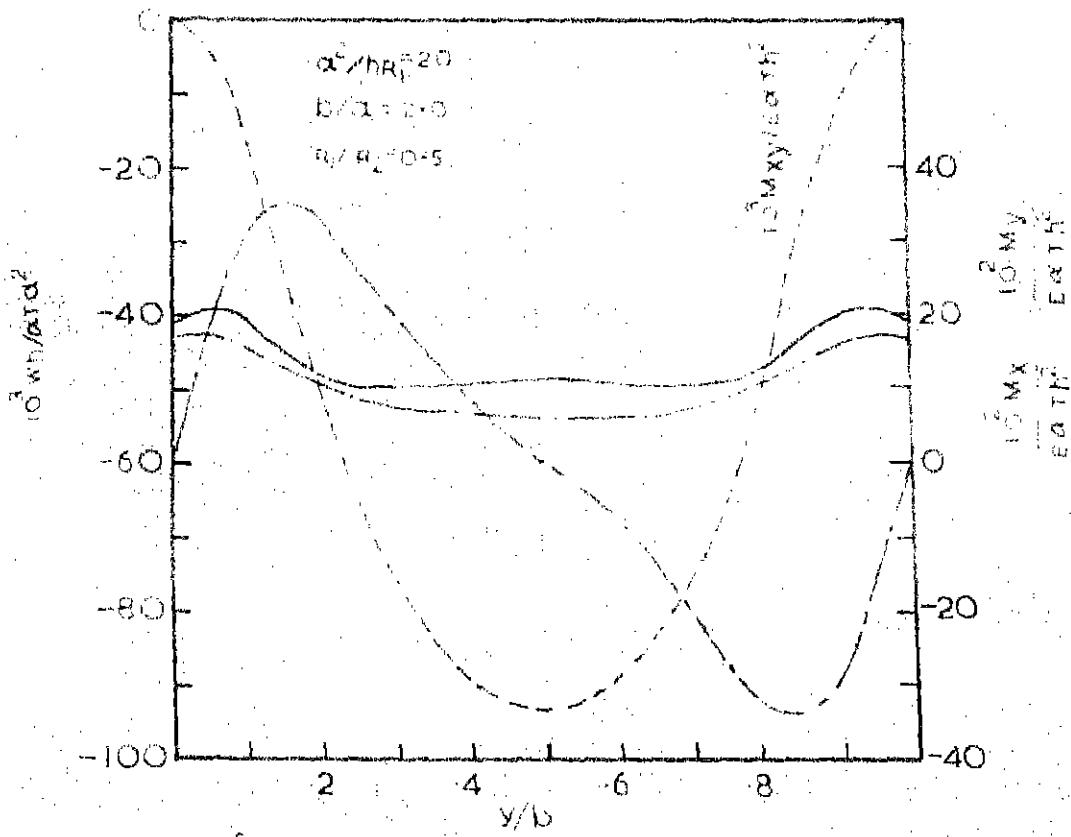
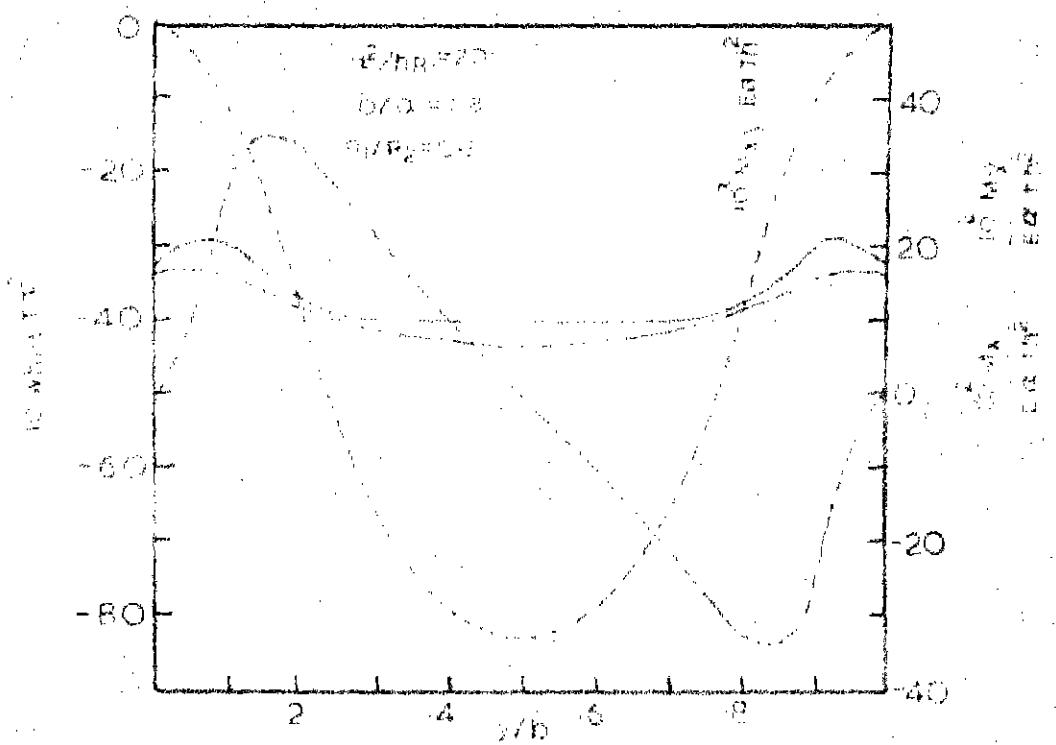
TWO EDGES CLAMMED
TWO EDGES SIMPLY SUPPORTED



10^3 Wh/ft^2 $10^3 \text{ My/ft}^2 \text{ E & T}$ $10^3 \text{ Mx/ft}^2 \text{ E & T}$ $10^3 \text{ Mxy/ft}^2 \text{ E & T}$

TWO EDGES CLAMPED

TWO EDGES SIMPLY SUPPORTED



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